OPTIMAL DESIGN OF THIN AND THICK WALLED CIVIL ENGINEERING STRUCTURES

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CERTIFICATE

This is to certify that the thesis "Optimal Design of Thin and Thick Walled Civil Engineering Structures" submitted by Shri A.V. Keskar in partial fulfilment of the requirements for the degree of Doctor of Philosophy of the Indian Institute of Technology, Kanpur, is a record of bonafide research work carried out by him under my supervision and guidance. The work embodied in this thesis has not been submitted elsewhere for a degree.

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LIST OF SYMBOLS

A	= cross-sectional area of a thin-walled member or a cantilever retaining wall
$[A]$	= matrix used to define a general eigenvalue problem
A_{s1}, A_{s2}	= main and distribution steel respectively in the stem of a cantilever retaining wall
A_{s3}, A_{s4}	= main and distribution steel respectively in the toe of a cantilever retaining wall
A_{s5}, A_{s6}	= main and distribution steel respectively in the heel of a cantilever retaining wall
a_x, a_y	= co-ordinates of the shear centre of a thin-walled section
$[B]$	= matrix used to define a general eigenvalue problem
	= modified matrix used in a standard eigenvalue problem
b	= top width of the cantilever retaining wall
b_x	= distance of the centroid from the bottom of a thin-walled section
bf_1	= top flange width of an I-section
bf_{1max}, bf_{1min}	= maximum and minimum values of top flange width of an I-section
bf_2	= bottom flange width of an I-section
bf_{2max}, bf_{2min}	= maximum and minimum values of bottom flange width of an I-section
C_s	= critical stress in the compression element
c	= a constant
	= reduction factor for the penalty parameter
	= cohesion of soil

$c_1, c_2, \ldots, c_8$ = constants of integration

c_b = steel cover in the base slab of a cantilever retaining wall

c_s = steel cover in the stem of a cantilever retaining wall

D = $2(1 - \mathrm{ch}\ \kappa) + \kappa\ \mathrm{sh}\ \kappa$

d = characteristic dimension of a cross-section

E = elastic (Young's) modulus

e_y = lateral eccentricity of transverse loading with respect to the web plane of an I-section

$F_1, F_2, \ldots, F_4$ = external forces on a cantilever retaining wall

$\{F_i\}$ = vector of external equivalent nodal loads

$F_{r1}, F_{r2}, \ldots, F_{r4}$ = internal resisting forces in a cantilever retaining wall

FS = factor of safety for the axial load in a column

FS_o = factor of safety against overturning of a cantilever retaining wall

FS_s = factor of safety against sliding of a cantilever retaining wall

f = objective function

f_i = objective functions in a multiple objective function problem

= function values in the Powell's method

G = shear modulus

g_j = constraint relations

$[g_{ij}]$ = stability matrix

$$\left.\begin{array}{l}[g^{xx}] \\ [g^{yy}] \\ [g^{\varphi\varphi}] \\ [g^{x\varphi}],\ [g^{\varphi x}] \\ [g^{y\varphi}],\ [g^{\varphi y}]\end{array}\right\} = \text{submatrices of } [g_{ij}]$$

H	= height of a column
ΣH	= summation of horizontal forces on the base of a cantilever retaining wall
$[H_i]$	= Hessian matrix
h	= total height of a cantilever retaining wall
h_1	= stem height of a cantilever retaining wall
hw	= web depth of an I-section
hw_{max}, hw_{min}	= maximum and minimum values of web depth of an I-section
I_{xx}, I_{yy}	= principal moments of inertia of a thin-walled section
I_{yy1}, I_{yy2}	= moments of inertia of the top and the bottom flange respectively of an I-section
$I_{\Omega\Omega}$	= principal sectorial moment of inertia of a thin-walled section
$[I]$	= identity matrix
i	= inclination of surcharge to the horizontal
	= imaginery unit
	= counter in an iterative process
J_o	= conjunct or concomitant, the sum of all integrated terms
J_1	= subset of constraints, active at the optimal point
J_2	= subset of constraints, inactive at the optimal point

j = constant in the reinforced concrete design

= counter in an iterative process

K = polar moment of inertia of a thin-walled section

= soil parameter, $(1 - \sin\emptyset)/(1 + \sin\emptyset)$

k^2 = ratio, torsional rigidity (GK)/warping rigidity $(EI_{\Omega\Omega})$

k = counter in an iterative process

$[k_{ij}]$ = stiffness matrix

$[k^{xx}], [k^{yy}], [k^{\varphi\varphi}]$ = submatrices of $[k_{ij}]$

L = length of a thin-walled member

= effective length of a member

= span of a beam

$[L]$ = lower triangular matrix

l = length of a discrete element

= length of a shell

l_i = equality constraints

$M_1, M_2, \ldots, M_4$ = external bending moments in a cantilever retaining wall

$M_{r1}, M_{r2}, \ldots, M_{r4}$ = internal resisting moments in a cantilever retaining wall

M_x, M_y = transverse moments in the principal planes of a thin-walled member

M_Ω = bimoment in a thin-walled member

$[M_i]$ = matrix generated in the Davidon-Fletcher-Powell method

5.9.1.9 Minimum thickness of the heel slab to resist shearing force

The external shearing force at section B-B [Figure 5.3(a)] is

$$F_4 = W_4 + w_c x_2 (x_3 - x_4 - x_1) - \frac{p}{2} \frac{(x_3 - x_4 - x_1)^2}{x_3}$$

The internal resisting force in reinforced concrete is

$$F_{r4} = F_{r3}$$

Therefore,

$$g_9(\bar{X}) = \frac{F_4}{F_{r4}} - 1 \qquad \text{... 5.9(i)}$$

5.9.1.10 Minimum thickness of the heel slab to resist bending moment

The external bending moment at section B-B [Figure 5.3(a)] is

$$M_4 = [W_4 + w_c x_2 (x_3 - x_4 - x_1)] \cdot \frac{(x_3 - x_4 - x_1)}{2}$$

$$- \frac{p}{6} \cdot \frac{(x_3 - x_4 - x_1)^3}{x_3}$$

while the internal moment of resistance of the balanced section is

$$M_{r4} = M_{r3}$$

Therefore

$$g_{10} = \frac{M_4}{M_{r4}} - 1 \qquad \text{... 5.9(j)}$$

m	= number of inequality constraints
	= modular ratio in the reinforced concrete design
	= direction in which maximum decrease in the function value takes place in the Powell's method
m_t	= distributed twisting moments on a member
m_x, m_y	= distributed bending moments in the principal planes of a member
m_Ω	= distributed bimoment on a member
$[m_{ij}]$	= mass matrix
$[m^{xx}]$, $[m^{yy}]$, $[m^{\varphi\varphi}]$, $[m^{x\varphi}]$, $[m^{\varphi x}]$, $[m^{y\varphi}]$, $[m^{\varphi y}]$	= submatrices of $[m_{ij}]$
N_z	= longitudinal force in a thin-walled member
$[N_i]$	= matrix generated in the Davidon-Fletcher-Powell method
n	= number of design variables
	= number of homogeneous equations in an eigenvalue problem
	= total number of the generalized nodal displacements for an element
	= unit normal pressure on the soil
ndf	= degree of freedom for a system
P	= axial load on a thin-walled member
	= resultant of lateral earth pressure on the back of a retaining wall
P'	= resultant of lateral earth pressure on the face of a retaining wall

5.11 OBJECTIVE FUNCTION

If r_s represents the cost of steel in rupees per unit weight of steel and r_c, the cost of concrete in rupees per unit volume of concrete, the objective function, being the material cost per running metre of wall, can be expressed as

$$f(\bar{X}) = r_s W_{st} + r_c V_c \qquad \ldots 5.27$$

5.12 DATA FOR THE PROBLEM

5.12.1 Soil Properties

Five different classes of soil are considered and they are: gravel, dense sand, loose sand, silt and clay. The first three types belong to the cohesionless variety while the other two to the cohesive one. Referring to IS : 1498 (1970), Terzaghi (1960), Tschebotarioff (1951) and Huntington (1967), *representative* values are selected for the unit weight (w_s), angle of internal friction (ϕ), permissible bearing capacity (p_{per}) and cohesion (c) for the soil retained. These values are given in Table 5.1.

5.12.2 Preassigned Parameters

Three different values are selected for the top width b, namely 20 cm, 25 cm and 30 cm. The range of overall height h chosen is from 3.0 to 6.0 m.

p_{bc} = permissible bending stress in compression

p_c = allowable stress in axial compression

P_{cr} = critical load on a compression member

P_x, P_y = Euler's flexural buckling loads in the principal planes of a member

P_φ = torsional buckling load for a member

p = total number of constraints

= maximum value of foundation pressure on the base of a retaining wall

p_e = maximum value of lateral earth pressure on the back of a retaining wall

p_{per} = permissible bearing capacity of the foundation

p_x, p_y, p_z = distributed loads on a member in the co-ordinate directions

Q = a constant in the reinforced concrete design

$\{Q_i\}$ = vector of generalized nodal forces

= vector generated in Davidon-Fletcher-Powell method

$\{Q^A\},\{Q^B\},\{Q^C\}$ = subvectors of generalized forces at nodes A, B and C

$q(z,t)$ = displacement function

$q_i(t)$ = nodal amplitudes

$\{q_i\}$ = vector of generalized nodal displacements

$\{q_1\}$ = modified displacement vector in a standard eigenvalue problem

$\{q^A\},\{q^B\},\{q^C\}$ = subvectors of generalized displacements at nodes A, B and C of a member

$\left.\begin{array}{l}\{q^x\},\ \{q^y\}\\ \{q^z\},\ \{q^\varphi\}\end{array}\right\}$ = subvectors of generalized displacements corresponding to flexure in principal planes, axial stressing and flexural torsion respectively at a node in a member

Following such a rational approach, the optimal solution obtained is obviously a local minimum. For some cases, such optimal solutions were successfully checked for the Kuhn-Tucker conditions for the local minimum. However, afterwards it was felt unnecessary to check every optimal solution for these conditions and as such they are not incorporated in the programmes OPT2, OPT3 and OPT4. The results obtained for the various cases in the programme OPT2 are shown in Tables 6.2 through 6.37. The results obtained from the programmes OPT3 and OPT4 are graphically represented in Figures 6.1 through 6.4.

6.1.2 Programme OPT1

A few optimal solutions obtained from this programmes were also successfully checked for the Kuhn-Tucker conditions for the local minimum. Later on, however, these conditions have not been included in the programme OPT1, as it was not found necessary. The results have been graphically represented in Figures 6.5 through 6.7.

6.2 DISCUSSION OF RESULTS

The results are discussed in the order in which the various optimization programmes have been handled in Chapters 3 through 5.

R = resultant foundation pressure on the base of a retaining wall

$\{R\}$ = vector of fixed end reactions for a member

r^2 = cross-sectional parameter

r = penalty parameter

r_c = unit cost of concrete

r_s = unit cost of steel

S = shearing strength of soil

$\bar{S}_i, \bar{S} = \{S_i\}$ = direction vector

SR_1 = slenderness ratio of compression flange of an I-section

SR_2 = slenderness ratio of a column

T = total twisting moment applied at a section

T_s = St. Venant's torque at a section

T_Ω = warping torque at a section

tf_1 = thickness of top flange of an I-section

tf_{1max}, tf_{1min} = maximum and minimum values of top flange thickness of an I-section

tf_2 = thickness of bottom flange of an I-section

tf_{2max}, tf_{2min} = maximum and minimum values of bottom flange thickness of an I-section

tw = thickness of web of an I-section

tw_{max}, tw_{min} = maximum and minimum values of web thickness of an I-section

V = potential energy of a system

ΣV = summation of vertical forces on the base of a cantilever retaining wall

V_c = volume of concrete

For the top width values of 0.25 and 0.30 m, the variation in the optimum cost is from Rs. 502.18-1592.42 and Rs. 563.29-1656.82 respectively, the optimal design being governed by the lower bound on the stem root thickness.

6.2.4.2 Dense sand

The lower bound on the stem root thickness continues to be the active constraint for heights from 3.0 to 4.5 m, the top width being 0.20 m. The corresponding variation in the optimum cost is from Rs. 464.95 to Rs. 893.75. For heights above 4.5 m, the active constraint is on the safety against failure of the wall by sliding on the foundation. The optimum cost variation is from Rs. 1112.62 to Rs. 1619.25 as the height changes from 5.0 to 6.0 m.

When the top width has values of 0.25 and 0.30 m, the optimum cost ranges from Rs. 521.56-1670.26 and Rs. 569.13-1694.29 respectively as the wall height varies from 3.0 to 6.0 m. In most of these cases, the lower bound on the stem root thickness governs the optimal design.

6.2.4.3 Loose sand

For a top width of 0.20 m, the optimum cost ranges from Rs. 544.19 to Rs. 2156.26 as the wall height varies from 3.0 to 6.0 m. The lower bound on the stem root thickness behaves as an active constraint for wall heights upto 4.0 m, after which the active constraint is on the safety against

For larger heights, a change over from the fixed to simple supports will substantially increase the cost but it may not do so for that from simple supports to a cantilever.

6.3.3 Optimization of Thin-Walled Sections under Dynamic Constraints

For a span of 3.0 m, as the support conditions are changed from fixed to simple, there is no increase in the optimum weight, when the bounds on the first natural vibrational frequency have a broad range; in case of a narrow range, the increase in the optimum weight is by 16.54%. For a span of 6.0 m, these increases are 74.90% and 23.22% respectively.

Hence, fixed beams are significantly economical compared to simple beams, for larger spans and subject to a broad range of bounds on the natural vibrational frequency; in case of smaller spans, the economy achieved may be marginal.

On the other hand, if the prescribed bounds have a narrow range, a small economy may be expected by choosing a fixed beam for small or large spans.

6.3.4 Optimization of a Cantilever Retaining Wall

6.3.4.1 Effect of top width

When the earth fill is gravel, for a height of 3.0 m, the optimum cost increases substantially by 23.95% as the top width changes from 0.20 to 0.30 m. However, this

V_x, V_y = transverse shear forces in the principal planes of a member

$V_{sd1}, V_{sd2}, V_{sd3}$ = volume of distribution steel in the stem, toe and heel respectively of a cantilever retaining wall

$V_{sm1}, V_{sm2}, V_{sm3}$ = volume of main steel in the stem, toe and heel respectively of a cantilever retaining wall

V_{st1} = volume of temperature reinforcement in a cantilever retaining wall

V_{ss}, V_{sh}, V_{st} = total volume of steel in the stem, heel and toe respectively of a cantilever retaining wall

W = weight of cantilever retaining wall

$W_1, W_2, \ldots, W_4$ = weights of different portions of a cantilever retaining wall

W_{st} = weight of steel in a cantilever retaining wall

w_c = unit weight of reinforced concrete

w_o = displacement component in z-direction

w_s = unit weight of soil

w_{st} = unit weight of steel

$\bar{X} = \{x_i\}$ = design vector

$$\bar{X}_i = \{X_i\} = \begin{Bmatrix} x_1^{(i)} \\ x_2^{(i)} \\ \vdots \\ x_n^{(i)} \end{Bmatrix} = \text{a design point}$$

$\bar{X}^* = \{X^*\}$ = $\{x_i^*\}$ = constrained optimal point

x_{imax}, x_{imin} = maximum and minimum values of a design variable x_i

y_1, y_2 = spacing of main and distribution steel respectively in the stem,

Y_3, Y_4 = spacing of main and distribution steel respectively in the toe,

Y_5, Y_6 = spacing of main and distribution steel respectively in the heel of a cantilever retaining wall

z = eccentricity of the foundation pressure resultant with respect to the base centre

α_i = relative weightages assigned to various functions in a multiobjective function

β_x, β_y = cross-sectional parameters

$\gamma_i(z)$ = displacement modes

Δ = maximum difference of function values in the Powell's method

δ = thickness of a cross-section

δ, δ' = obliquity of soil pressure resultants P and P' respectively

η = displacement component in y-direction

κ = kl, a constant for a thin-walled member

λ = coefficient used to define a convex function

= eigenvalue

λ_j = Lagrange multipliers

λ^*, λ_i^* = optimum step length

λ_{min} = smallest eigenvalue

ξ = displacement component in x-direction

ξ_{per} = permissible value of deflection in a beam

ρ = mass per unit volume

σ = normal stress at a point

$\sigma^{(1)}, \sigma^{(2)}$ = normal stress at points (1), (2) etc. in a cross-section

σ_i^{max} = maximum value of normal stress at the ith node

Table 6.6

Optimal Sectional Dimensions

Loading: uniformly distributed

Intensity: 3000 kg/m ; Lateral Eccentricity: nil

Span in m	Fixed Supports								Simple Supports							
	Optimal dimensions in mm						Optimum weight in kg/m	Active constraint/s	Optimal dimensions in mm						Optimum weight in kg/m	Active constraint/s
	bf_1	tf_1	bf_2	tf_2	hw	tw			bf_1	tf_1	bf_2	tf_2	hw	tw		
3.0	90.4	6.9	90.4	6.9	176.0	5.1	16.87	g_{13}^*	100.0	7.8	100.0	7.8	209.0	5.5	21.29	g_{13}
4.0	100.0	8.6	100.0	8.6	225.0	5.8	23.75	g_{13}	129.8	8.7	129.8	8.7	258.0	6.2	30.37	g_{13}
5.0	132.8	8.8	132.8	8.8	263.0	6.2	31.06	g_{13}	155.4	9.5	155.4	9.5	309.0	6.8	39.82	g_{13}
6.0	152.4	9.5	152.4	9.5	304.0	6.7	38.77	g_{13}	165.0	11.4	165.0	11.4	351.0	7.4	50.00	g_{13}

*Maximum normal stress in the beam.

Table 6.16

Optimal Sectional Dimensions

Loading: linearly varying

Intensity: 1000-3000 kg/m ; Lateral Eccentricity: nil

Span in m	Fixed Supports								Simple Supports							
	Optimal dimensions in mm						Optimum weight in kg/m	Active constraint/s	Optimal dimensions in mm						Optimum weight in kg/m	Active constraint/s
	bf_1	tf_1	bf_2	tf_2	hw	tw			bf_1	tf_1	bf_2	tf_2	hw	tw		
3.0	80.0	6.8	80.0	6.8	150.0	4.8	14.19	g_{13}^{*}	90.4	6.9	90.4	6.9	176.0	5.1	16.87	g_{13}
4.0	99.2	7.3	99.2	7.3	198.0	5.4	19.68	g_{13}	100.0	8.6	100.0	8.6	225.0	5.8	23.75	g_{13}
5.0	114.0	8.7	114.0	8.7	239.0	6.0	26.70	g_{13}	134.6	8.8	134.6	8.8	266.0	6.3	31.67	g_{15}^{**}
6.0	140.8	8.8	140.8	8.8	277.0	6.4	33.50	g_{13}	156.0	9.6	156.0	9.6	310.0	6.8	40.01	g_{15}

*Maximum normal stress in the beam.

**Maximum deflection in the beam.

σ_{cr}	= critical value of normal stress in a member
σ_{per}	= permissible value of normal stress
σ_{bcr}	= critical value of buckling stress in a column
σ_{bw}	= critical value of compressive stress in a column at working loads
σ_{bper}	= permissible value of compressive stress in a column
σ_{cb}	= permissible compressive stress in bending for concrete
σ_{st}	= permissible stress in tension for steel
τ	= shear stress at a point
$\tau^{(1)}, \tau^{(2)}, \tau^{(3)}$	= shear stress at points (1), (2), (3) etc. in a cross-section
τ_i^{max}	= maximum value of shear stress at ith node
τ_c	= permissible shear stress in concrete
τ_{cr}	= critical value of shear stress in a member
τ_{per}	= permissible value of shear stress
$\emptyset$	= penalty function
	= angle of internal friction for soil
φ	= rotation of member about the longitudinal axis
Ω	= normalized sectorial co-ordinate
ω	= natural frequency of transverse vibration
ω_1	= value of natural frequency corresponding to the first mode of transverse vibrations
ω_{max}, ω_{min}	= upper and lower bounds on the natural frequency of transverse vibration

$\bar{\nabla}$ = vector operator $\begin{Bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{Bmatrix}$

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Table 6.26

Optimal Sectional Dimensions

Loading: uniformly distributed

Intensity: 1000 kg/m ; Lateral Eccentricity: 0.1 bf_1

(from Table 6.3)

Span in m	Cantilever							
	Optimal dimensions in mm						Optimum weight in kg/m	Active constraint /s
	bf_1	tf_1	bf_2	tf_2	hw	tw		
1.0	51.0	6.4	51.0	6.4	101.0	4.0	8.31	none
1.5	79.0	6.7	79.0	6.7	145.0	4.7	13.73	g_{13}^*
2.0	97.6	7.2	97.6	7.2	194.0	5.3	19.17	g_{13}

*Maximum normal stress in the beam.

SYNOPSIS

Engineering is the activity through which designs for material objects are produced. In the field of Civil Engineering, a structural engineer is confronted with the problem of designing a wide range of structures, such as massive dams, retaining walls etc. (thick structures) at one end of the spectrum and thin-walled component members like rolled steel joists, columns etc. at the other end.

It is well-known that the primary function of a structure is to resist and/or transmit forces through solid matter. To increase the *efficacy* of a structure, the structural designer should not only work out a satisfactory design but also should endeavour to see that it is an optimal one. When a range of designs exists within a selected design concept, the optimal design is chosen in such a way that it not only satisfies all the limitations and restrictions placed on it, but it is best in some sense, when compared to other possible designs in the set.

Although, the basic principles of structural analysis are same for all structures, the approaches for thick and thin structures are at variance. Even the design objectives in these two classes of structures are different. In other words, the set of constraints to be considered in the design process are to be selectively chosen. However,

Table 6.36

Optimal Sectional Dimensions

Loading: uniformly distributed

Intensity: 3000 kg/m ; Lateral Eccentricity: 0.1 bf_1

(from Table 6.7)

Span in m	Cantilever							
	Optimal dimensions in mm						Optimum weight in kg/m	Active constra-int/s
	bf_1	tf_1	bf_2	tf_2	hw	tw		
1.0	90.0	6.9	90.0	6.9	175.0	5.1	16.76	g_{13}^{*}
1.5	116.0	8.7	116.0	8.7	241.0	6.0	27.12	g_{13}
2.0	151.8	9.5	151.8	9.5	303.0	6.7	38.55	g_{13}

*Maximum normal stress in the beam.

the tools available for the structural optimization, such as mathematical programming, optimal synthesis etc. can be applied with equal ease to both these categories of structures.

A reinforced concrete cantilever retaining wall is chosen to represent the thick-walled class. In such structures, the material cost being the main objective, optimal designs are obtained for minimum cost for five types of soil commonly met with in practice.

As rolled steel sections (I-shape) are quite commonly used in Civil Engineering construction, structural I-sections are chosen to represent the thin-walled category. In such a case, weight being the criterion, optimal designs of beams and columns of I-section are obtained subject to various types of loading and boundary conditions, under the *static*, the *stability* and the *dynamic* constraints.

Under the optimization subject to static constraints, the loading cases considered are all of practical significance. The case of concentrated loads acting at selected points on a beam is considered, for example. Such type of loads are quite common in the form of reactions transferred on the main girder through a cross-girder system. A similar other case of practical importance is of a laterally eccentric distributed load that produces significant torsion in the beam. Normally a section with unequal flange widths and thicknesses is expected to perform better in such a loading

Carroll, C.W.	1961	The created response surface technique for optimizing non-linear restraint systems, Oper. Res., Vol. 9, pp. 169-84
Chajes, A. and Winter, G.	1965	Torsional-flexural buckling of thin walled members, J. Struct. Div., Proc. ASCE, Vol. 91, No. ST4, pp. 103-29
Chajes, A.	1974	Principles of structural stability theory, Prentice Hall, Englewood Cliffs, New Jersey
Chang, Pin Yu, Thasantorn, C. and Pilkey, W.	1975	Restrained warping stresses in thin walled open sections, J. Struct. Div., Proc. ASCE, Vol. 101, No. ST11, pp. 2467-71
Chilver, A.H., Ed. by	1967	Thin walled structures, John Wiley, New York
Conway, H.D.	1972	A simple formula for the maximum stress in a twisted angle or channel, Int. J. Mech. Sci., Vol. 14, No. 9, pp. 615-17
Dahlquist, G. and Bjorck, A.	1974	Numerical methods, Prentice-Hall, Englewood Cliffs, New Jersey
Davidon, W.C.	1959	Variable metric method for minimization, Argonne National Laboratory Report No. ANL-5990
Dawe, D.G.	1969	Application of the discrete element method to the buckling analysis of rectangular plates under arbitrary member loading, Aero. Quart., Vol. XX, Part 2, pp. 114-28
Desai, C.S. and Abel, J.F.	1972	Introduction to the finite element method, Van Nostrand Reinhold, New York
Dooley, J.F.	1967	The torsional deformation of columns of monosymmetric I-sections with restrained axis of twist, under doubly eccentric load, Int. J. Mech. Sci., Vol. 9, No. 9, pp. 585-604

situation; however, it is interesting to note that the actual optimal section possess *equal* values of these dimensions.

In both the problems of optimization of thick- and thin-walled structures, the constraints used are those specified by the Indian Codes of Practice. In the latter problem, the significant feature is that, the analysis is carried out by the discrete element technique incorporating Vlasov's thin-walled beam theory. The consistent matrix formulation, thus generated, is quite versatile so that thin-walled beams, having any shape of cross-section formed by an assemblage of plate elements, can be analysed by this technique. By suitably imposing the boundary conditions at the intermediate nodes of the discretized structure, the optimization problem can be extended to include continuous beams as well.

Lastly, the discrete element analysis considers the effect of warping of the section, an attempt *never* made so far, for obtaining optimal designs of thin-walled sections. The results thus obtained are, therefore, more rational and could be utilized in practice.

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1. INTRODUCTION

1.1 GENERAL

A structure is an element or an assemblage of elements put together into a shape, to resist and/or transmit forces. It may also be looked upon as a system consisting of subsystems which in turn comprise many structural elements.

A structure is referred to as a planar or a space structure depending on whether or not the centre line of the elements forming the structure lie in a single plane. Depending on the shape (structural form) and nature of external and/or internal *influences* (forces), structures are called *rods*, *beams*, *trusses*, *frames*, *plates*, *shells* etc. All structures are invariably three dimensional, but depending on the relative ratios of sides, they are treated as *one-*, *two-* and *three-dimensional* continua; rods, beams, trusses, frames belong to the first category while plates shells etc. belong to the second. Whenever it is possible, three dimensional structures are reduced to lower dimensional category for purposes of analysis and design.

1.2 INFLUENCES (FORCES)

The environment in which a structure is situated, is usually idealized as a system or systems of *forces*. Forces caused by interactions between a structure and its environment are referred to as *external forces* and forces between

different parts of an isolated system are known as *internal forces*. The forces acting on a structure may be static or dynamic, internal or external, fixed (deterministic) or random (probabilistic) etc.

1.3 STRUCTURAL ANALYSIS

The purpose of structural analysis is to study or predict the response or behaviour of a loaded structure. It is based on four requirements namely, the *equilibrium*, the *condition of compatibility*, the *force-deformation relationship of the material* used and the *known boundary conditions*. If all the foregoing conditions are satisfied, the solution is exact. However, a solution may yield useful information insofar as to give a bound on the exact solution, if a few of the foregoing requirements are satisfied.

Depending on the range of mechanical behaviour considered, an analysis may be *broadly* classified as *elastic* or *plastic*. Elastic analysis predicts the behaviour at working or design loads while the plastic analysis does so at collapse. In this thesis, the elastic analysis is dealt with, exclusively.

1.4 STRUCTURAL DESIGN

The primary purpose of structural design is force transmission by solid matter with certain design objectives such as structural *safety*, *rigidity*, *serviceability*, or

geometric, functional and *aesthetic* requirements. Traditionally, a trial design is an educated human guess based on intuition, common sense, professional judgement and experience. Certain assumptions are made, a trial design worked out, tested by detailed examination (analysis), modified in the light of the results obtained and re-examined. This iterative process is continued until all the prescribed conditions *(constraints)* are satisfied. Thus, the design is treated as a process of trial and error, except when formulated as either a *direct design* or an *optimal design* problem.

1.5 STRUCTURAL DESIGN IN CIVIL ENGINEERING PROBLEMS

In the field of civil engineering, a structural engineer is confronted with the problem of designing a wide range of structures, such as massive dams, retaining walls etc. *(thick structures)* at one end of the spectrum and thin-walled component members like rolled steel joists, columns etc. *(thin structures)* at the other end.

Although the basic principles of structural analysis are same for all structures, the approaches for thick and thin structures are at variance. In structures like dams, retaining walls etc. there exists a significant interaction between the surrounding medium and the structure. As such, the internal forces between the idealized environment and the structure have to be considered in its design. In thin-walled structures,

the surrounding medium has no influence insofar as its structural design is concerned.

The term 'failure' of a structure may mean fracture, fatigue, instability or total collapse. The *failure* may occur at a localized portion in the structure *(local failure)* or the gross structure may fail *(overall failure)*. In a thick-walled structure, it is sufficient to analyse it for the overall failure. On the other hand, due to the 'size effect', one has to investigate for local failure as well in a thin-walled structure, which makes its analysis more involved.

A designer should not only prepare a working design of a structure, but should also endeavour to see that it is an optimum one, keeping a certain objective in mind. Even the *design objectives* in these classes of structures are different, usually. Thick-walled structures in civil engineering are invariably of masonry or plain or reinforced concrete. These are usually cast-in-situ wherein the *material cost* is bulk of the total cost. The optimal design of such structures would be aimed at the minimal cost.

Thin-walled structures are usually erected from prefabricated metallic component members that are assembled at site. Transportation and handling being an important operation involved in such a case, one would prefer to cut down the *weight* of the structural component members to a minimum. Weight is, therefore, chosen as the design objective in such a class of structures.

1.6 MECHANICAL BEHAVIOUR OF ENGINEERING STRUCTURES

Structures respond in different ways when subjected to external influences depending on their geometric characteristics and on the mechanical and other properties of the materials used. The mechanical properties of a material may be represented by determining a functional relation between the stresses and strains or more generally between *generalized stresses* and *generalized strains*.

The actual behaviour of a structural material is idealized by making certain reasonable assumptions. A typical relation between generalized stress and generalized strain is shown in Figure 1.1(a).

For most materials encountered in engineering practice, it is practically impossible to describe the entire stress-strain curve with a *single* mathematical expression. Hence the stress-strain curve is idealized as a piecewise-continuous one. Depending upon the aspects that are most important in a given problem, only that part of the curve pertaining to the range of material behaviour under consideration is idealized.

Steel and concrete are the most common materials used in civil engineering structures. Structural steel exhibits a marked plastic behaviour from the elastic one, the yield point separating the two. The stress-strain curve for such an ***elastic-perfectly plastic*** material can be idealized as shown in Figure 1.1(b).

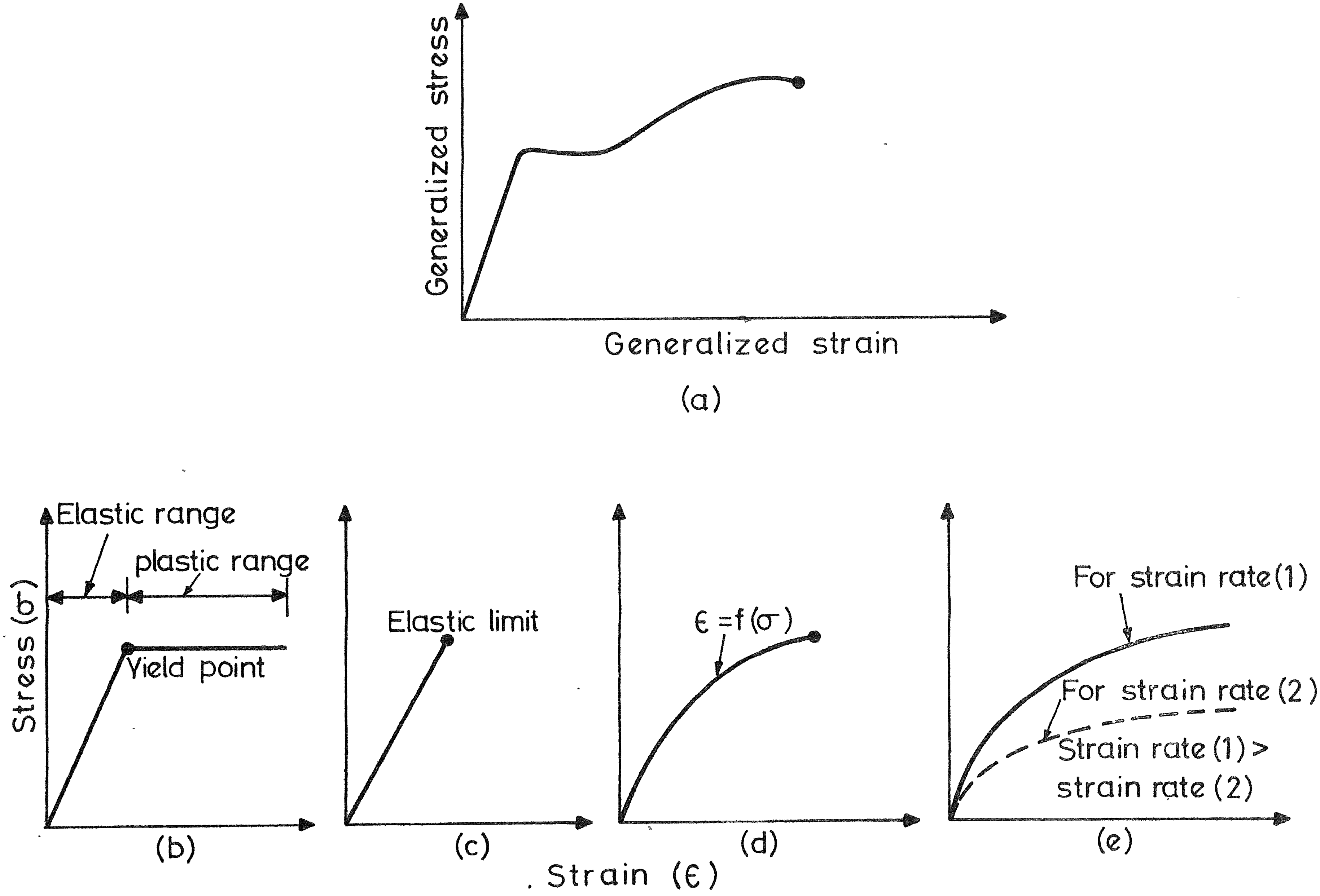


Fig.1.1 **Stress strain curves.**

The stress-strain relationship for a material in the elastic range may be either *linear* [Figure 1.1(c)] or *non-linear* [Figure 1.1(d)] and the material is then referred to as linearly- or non-linearly-elastic. Structural steel can be reasonably approximated to follow a linear elastic behaviour, although strictly speaking, it is a non-linear one.

Concrete exhibits a time-dependent behaviour and as such, its stress-strain characteristic changes according to the applied strain rate. Such behaviour is referred to as *visco-elastic* typically represented in Figure 1.1(e).

Concrete is heterogeneous in composition. The constituent crystals in the micro-structure of the structural effect are randomly oriented. However, for all practical purposes in structural analysis, both concrete and steel are assumed to be *homogeneous*, *isotropic* and *linearly elastic*.

In addition to the structural material properties, a structural designer has to take cognizance of characteristics of the surrounding medium. For example, in a structure like a retaining wall, this medium being the soil retained, such physical properties as the *unit weight*, *porosity*, *cohesion*, *shear strength*, *angle of internal friction*, *angle of repose*, *permeability*, *bearing capacity* etc. are to be taken into account.

1.7 FORCE-DEFORMATION RELATIONSHIP IN A STRUCTURE

A structure subjected to the action of external forces gives rise to the internal forces *(stresses)* that

deform the structure and eventually maintain its equilibrium. The deformations that usually include translational and rotational *displacements* are generally very small. Hence the analysis of internal forces may be based on the dimensions of the *undeformed shape* without appreciable error. This approach to the solution of structural systems is called the *small deflection theory*, which is used in the present analysis. Within the limitation of small deflection, structural systems are usually stressed below the elastic limit of the material. Hence the small deflection theory is also known as the *elastic theory*.

If the stress-strain relation of the material is linearly elastic, the equations representing the force deformation relations are also *linear functions* of internal forces. Such structural systems are called *linear*, whose important characteristic is that the *principle of superposition* can be used for their solutions, as is true in the present case.

1.8 STRUCTURAL OPTIMIZATION

1.8.1 General

Engineering is the activity through which designs for material objects are produced. In many design problems, there are several possible alternative *design concepts*, for example, a girder for a highway bridge can be concrete or steel, and once material is chosen, several approaches to

using it are available. The goal of the design process may range from providing a practicable solution to a problem where none is previously known, to improving on or replacing an existing design. Adopting this point of view that a range of design exists within a selected design concept, the concept of the *optimum design* is initiated because of the presumption that these values are to be chosen in such a way that the design will be the one that satisfies all the limitations and restrictions placed on it and is best in some sense.

1.8.2 Optimization Problem

An optimization problem is usually concerned with the *extremization* of a function (functional) of several variables (functions) within prescribed *constraints*. Since any maximization technique can be transformed into a minimization technique and vice versa *(duality)*, without loss of generality, optimization can be taken to mean *minimization*. There is no single method available for solving all optimization problems efficiently. Hence a number of optimization methods have been developed for solving different types of optimization problems. The optimum seeking methods are also known as *mathematical programming techniques* and are generally studied as a part of the operations research branch in mathematics, that deals with the general problem of optimality.

1.9 MATHEMATICAL PROGRAMMING TECHNIQUES

Table 1.1 gives the various mathematical programming techniques along with the other well-defined areas of operations research.

The mathematical programming techniques are useful in finding the minimum or maximum of a function of several variables subject to limitations *(constraints)* that are expressed as *equalities* or *inequalities*. The stochastic process techniques can be used to analyse problems which are described by a set of *random variables* having known *probability distribution*. The statistical methods enable one to analyse the experimental data and build empirical models to obtain the most accurate representation of the physical situation.

In the procedure characterized as mathematical programming formulations, the representation of inequality constraints is of critical importance since this permits the design to be identified as one in which not all structural elements are subject to limiting conditions under specified loads. Also, in such formulations, the orientation towards many variable problems fits quite well with the recent trend in analysis towards *discrete element* representations that require large order systems. Hence,the scope of the present thesis is confined to the application of one of the mathematical programming techniques using the discrete element analysis of the idealized structure.

Table 1.1

Methods of Operations Research

Methematical Programming Techniques	Stochastic Process Techniques	Statistical Methods
(i) Calculus methods	(i) Statistical decision theory	(i) Regression analysis
(ii) Calculus of variations	(ii) Markov processes	(ii) Cluster analysis, pattern recognition
(iii) Non-linear programming	(iii) Queuing theory	(iii) Design of experiments
(iv) Geometric programming	(iv) Simulation methods	(iv) Discriminate analysis (factor analysis)
(v) Quadratic programming	(v) Reliability theory	
(vi) Linear programming		
(vii) Dynamic programming		
(viii) Integer programming		
(ix) Stochastic programming		
(x) Network methods: CPM and PERT		
(xi) Game theory		

1.10 HISTORICAL DEVELOPMENT

The existence of optimization methods can be traced back to the days of Newton, Lagrange and Cauchy. The development of *differential calculus methods of optimization* was possible because of the contributions of Newton and Leibnitz to calculus. The foundations of the *calculus of variations* were laid by Bernoulli, Euler, Lagrange and Weirstrass. The method of optimization for *constrained minimization* problems which involves the addition of unknown *multipliers* was invented by Lagrange. Cauchy made the first application of the steepest descent method to solve *unconstrained minimization* problems. In spite of these early contributions, very little progress was made until the middle of the twentieth century, when the high speed digital computers made the implementation of the optimization procedures possible and stimulated further research on new methods.

In 1947, Dantzig developed the simplex method for *linear programming* problems.

The principle of optimality was established by Bellman in 1957, for *dynamic programming* problems which paved the way for development of the methods of constrained optimization.

Duffin, Zener and Peterson, in 1967, developed *geometric programming*. Avriel and Williams, in 1970, extended this method to include any rational function of posynomial

terms and called the method as complementary geometric programming.

Gomory, in 1960, did pioneering work in *integer programming,* which is one of the most exciting and rapidly developing areas of optimization. The reason for this is that most of the real world applications fall under this category of problems.

Dantzig, in 1955, and Charnes and Cooper, in 1959, developed *stochastic programming* techniques and solved problems by assuming design parameters to be independent and normally distributed.

Network analysis methods are essentially management control techniques and were developed during the years 1957 and 1958.

The foundations of *game theory* were developed by von Newmann in 1928 and since then it has been applied to solve several mathematical, economics and military problems. Only during the last few years, game theory has been applied to solve some of the engineering design problems.

1.11 NON-LINEAR PROGRAMMING

If the optimization problem involves any of the objective or constraint functions as non-linear, the problem is called a *non-linear programming* problem. This is the most general programming problem and all other problems can be considered as special cases of this. Depending upon

whether the constraints exist or not in the problem, it is classified as a *constrained* or an *unconstrained* one.

Mathematical programming was first applied to structural optimization in the late 1950's. According to early contributions by Livesley (1956) and Pearson (1958), limit design was treated as a linear programming problem, while Schmit (1960) cast the elastic design as the more general non-linear programming problem.

1.11.1 Unconstrained Optimization

Major developments in the area of numerical methods of unconstrained optimization were made in the United Kingdom in the early 1960's.

Under one such category of methods called *direct search* methods, unconstrained minimization problems were initially handled by the *random search* and later by the *univariate* methods.

The *simplex* methods were introduced by Spendley, Hext and Himmsworth (1962). The original simplex triangle was modified and more effectively used by Nelder and Mead (1965) and by Box, Davies and Swann (1969).

Hooke and Jeeves (1961) initiated the *pattern search* methods. Rosenbrock (1960) independently introduced the method of *rotating co-ordinates,* which can be considered as the development of the former one. Powell (1964) extended the basic pattern search method. It is a very powerful and

the most widely accepted direct search method. Powell has suggested some modifications to facilitate the convergence of this method when applied to non-quadratic objective functions, which have made it more versatile.

Among other category of the unconstrained minimization methods, come the *descent* methods. One such type is the *steepest descent* method pioneered by Cauchy in the middle of nineteenth century. The method was used with moderate success on a wide variety of problems through the middle 1950's.

Fletcher and Reeves (1964) greatly improved the convergence characteristics by modifying it into a *conjugate gradient* method.

The oldest method for solving a set of non-linear equations is the Newton's method. Based on this, a class of methods called *quasi-Newton* methods was developed, since these can be regarded as approximations to the Newton's method in some sense. One such method is the *variable metric* method introduced first by Davidon (1959) and developed later on by Fletcher and Powell (1963).

1.11.2 Constrained Optimization

Many techniques have been developed for the solution of a constrained non-linear programming problem. All these techniques can be classified into two broad categories namely, the *direct* methods and the *indirect* methods. In the direct methods, the constraints are handled in an

explicit manner whereas in most of the indirect methods, the constrained problem is solved as a sequence of unconstrained minimization problems.

Among the direct methods, one such approach is by approximating the constraints. Kelly (1960) used the *cutting plane* method for solving *convex programming* problems.

Another method developed by Zoutendijk (1960) makes a successive use of *feasible directions* in obtaining the constrained minimum.

The most significant under the indirect methods are the *penalty function* methods. These are further categorized into the *interior penalty* and the *exterior penalty* function methods. They were successfully developed by Carrol (1961), Zangwill (1967) and Fiacco and McCormick (1963, 1964, 1968).

1.12 CIVIL ENGINEERING APPLICATIONS OF OPTIMIZATION

Optimization, in its broadest sense, can be applied to solve any engineering problem. In the field of civil engineering, optimization studies are useful to

(a) develop a fully automated design with the aid of high speed digital computers of structures like frames, foundations, bridges, towers, dams, retaining walls etc. for minimum cost.

(b) obtain minimum weight design of thin-walled structures for earthquake, wind and other types of random loading.

(c) develop optimal plastic design of structures.

(d) select a suitable configuration and topography for a water resources system and to design it for the maximum efficiency.

1.13 STATEMENT OF AN OPTIMIZATION PROBLEM

An optimization or a mathematical programming problem can be stated as follows

$$\text{find } \bar{X} = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} \text{ which minimizes } f(\bar{X})$$

subject to the constraints ... 1.1

$$g_j(\bar{X}) \leq 0, \quad j = 1, 2, \ldots, m$$

and, $$l_j(\bar{X}) = 0, \quad j = m+1, m+2, \ldots, p .$$

where $\bar{X}$ is an n-dimensional vector called the *design vector*, $f(\bar{X})$ is called the *objective function* and $g_j(\bar{X})$ and $l_j(\bar{X})$ are respectively the *inequality* and the *equality* constraints.

The number of variables 'n' and the number of constraints 'p' need not be related in any way. Thus, in a given mathematical programming problem, p could be less than, equal to or greater than n. In some problems, the value of p might be zero which means that there are no constraints in

the problem. Such type of problems are called *unconstrained* optimization problems. Those problems for which p is not equal to zero are known as *constrained* optimization problems.

1.13.1 Design Vector

Any engineering system or component is described by a set of quantities, some of which are treated as variables during the design process. In general, certain quantities are usually fixed at the outset and these are called *preassigned parameters*. All the other quantities are looked upon as *variables* in the design process and are called *design* or *decision variables* x_i, i = 1, 2, ..., n. The design variables are *collectively* represented as a design vector

$$\bar{X} = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} = \{x_i\}$$

The design variables of an optimum structural design problem may consist of the member sizes, parameters that describe the structural configuration and the mechanical or physical properties of the material as well as other quantifiable aspects of design. The topology (i.e. the pattern of connection of members in the mathematical model) of the complete structure is difficult to take into account although it is treated to a limited extent when the algorithm employed permits members to reach zero size.

If an n-dimensional Cartesian space with each co-ordinate axis representing a design variable x_i, $i = 1, 2, \ldots, n$, is considered, the space is called the *design variable space*. Each point in the n-dimensional design space is called a *design point* and it represents either a possible or an impossible solution to the design problem.

1.13.2 Design Constraints

In many practical problems, the design variables cannot be chosen arbitrarily; rather they have to satisfy certain specified functional and other requirements. The prescribed restrictions that must be satisfied in order to produce an acceptable design are collectively called *design constraints*. A constraint may take the form of a limitation imposed directly on a variable or group of variables *(explicit constraint)* or may represent a limitation on quantities whose dependence on the design variables cannot be stated directly *(implicit constraint)*.

The constraints which represent limitations on the behaviour or performance of the system are termed as *behavioural* or *functional* constraints. The constraints which represent physical limitations on the design variables like availability, fabricability and transportability are known as *geometric* or *side* constraints. The behavioural constraints in structural design are usually limitations on stresses or displacements. But they may also take the form of restrictions on such factors

as vibrational frequency or buckling strength. Behaviour constraints are generally implicit, although typically explicit behaviour constraints are given by formulae presented in design specifications.

In general, the given set of constraints may include the equality as well as inequality constraints. In theory, each equality constraint is an opportunity to remove a design variable from the optimization process and thereby reduce the number of dimensions of the problem. However, as the elimination procedure may be awkward and algebraically complicated, this approach is not always adopted.

The idea of an inequality constraint is of major importance in optimum structural design. If equality constraints only were stipulated in a design limited by stresses alone, all procedures would lead to *fully stressed design*. It has been proved that fully stressed designs are not necessarily *minimum weight designs* and for optimality, it is essential to permit designs in which not all stress constraints are satisfied identically i.e. inequality constraints.

In general, constraint equations are *non-linear* functions of the design variables when the behaviour of the structure is *elastic* and statically indeterminate although the analysis technology is *linear*.

1.13.3 Constraint Surface

Each constraint condition appears in design space as a surface representing the locus of design points which cause the constraint to be satisfied as an equality constraint.

For illustration, consider an optimization problem with only inequality constraints $g_j(\bar{X}) \leq 0$. The set of values of $\bar{X}$ that satisfy the equation $g_j(\bar{X}) = 0$ forms a *hypersurface* in the design space and is called a *constraint surface.*

It may be noted that this is an (n-1)-dimensional subspace, n being the number of design variables. The constraint surface divides the design space into two regions; one in which $g_j(\bar{X}) < 0$ and the other in which $g_j(\bar{X}) > 0$. Thus the points lying on the hypersurface will satisfy the constraints $g_j(\bar{X})$ *critically,* whereas the points lying in the region where $g_j(\bar{X}) > 0$ are *infeasible* or *unacceptable,* and the points lying in the region where $g_j(\bar{X}) < 0$ are *feasible* or *acceptablé.* The collection of all the constraint surfaces $g_j(\bar{X}) = 0$, $j = 1, 2, \ldots, m$ which separates the acceptable region is called the *composite constraint surface.* For continuous design variables, the constraint surface is in general *continuous,* and is *curved* for the elastic structural design of an indeterminate structure.

Figure 1.2 shows a hypothetical two-dimensional design space where the infeasible region is indicated by hatched lines. A design point which lies on one or more than

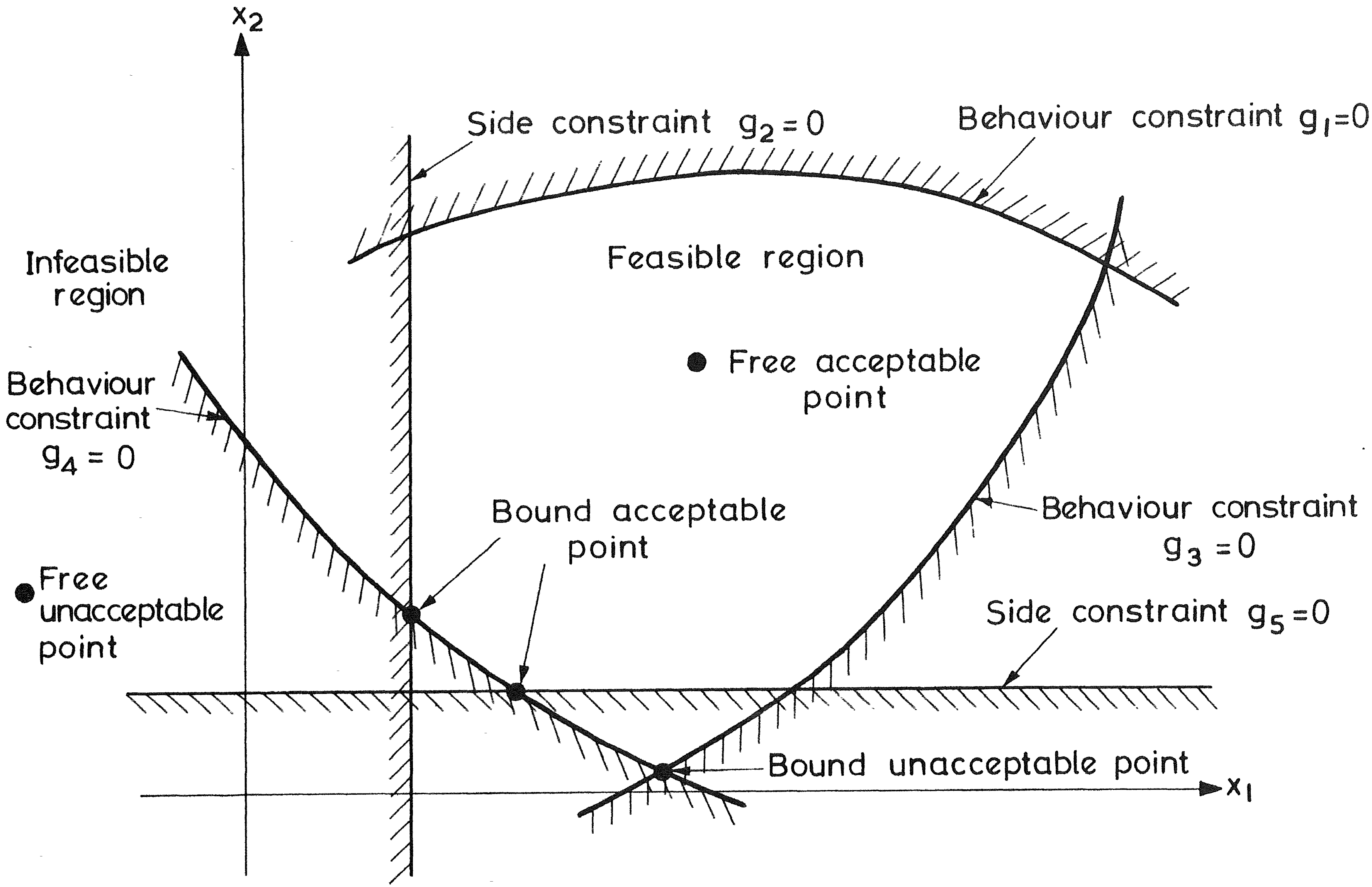


Fig. 1.2 Composite constraint surface.

one constraint surface is called a *bound* point and the associated constraint is called an *active* constraint. The design points which do not lie on any constraint surface are known as *free* points. Depending upon whether a particular design point belongs to the acceptable or unacceptable region, it can be identified as one of the following four types:

(a) free and acceptable point,

(b) free and unacceptable point,

(c) bound and acceptable point, and

(d) bound and unacceptable point.

All these four types of points are shown in Figure 1.2.

1.13.4 Objective Function

The conventional design procedures aim at finding an *acceptable* or adequate design, which merely satisfies the functional and other requirements of the problem. In general, there will be more than one acceptable design and the purpose of optimization is to choose the best one out of the many acceptable designs available. Thus a *criterion* has to be chosen that constitutes a basis for the selection of one of several alternative acceptable designs. The criterion with respect to which the design is optimized, i.e. whose least (or greatest) value is sought in an optimization procedure, when expressed as a *scalar function* of the design variables, is termed the *cost function*, *merit function* or *objective function*.

The objective function represents the most important *single* property of a design and its choice is governed by the nature of a problem. The objective function for minimization is generally taken as *weight* in aircraft and aerospace structural design problems. In civil engineering structural designs, it is usually the minimization of *cost*. The maximization of *mechanical efficiency* would be the obvious choice in mechanical engineering systems design problems.

In some situations, there may be more than one criterion to be satisfied simultaneously. One way to handle such a problem is to take the actual objective function as a weighted sum of the *multiple objective functions*. Thus if $f_1(\bar{X})$ and $f_2(\bar{X})$ are the two objective functions possible, the new objective function $f(\bar{X})$ may be constructed for optimization as

$$f(\bar{X}) = \alpha_1 f_1(\bar{X}) + \alpha_2 f_2(\bar{X}) \qquad \ldots 1.2$$

α_1, α_2 being the relative weightages assigned.

1.13.5 Objective Function Surfaces

The locus of all points satisfying $f(\bar{X}) = C =$ constant, forms a hypersurface in the design space and for each value of 'C' there corresponds a different member of a *family* of surfaces. These surfaces called the *objective function surfaces* are shown in a hypothetical two-dimensional design space in Figure 1.3.

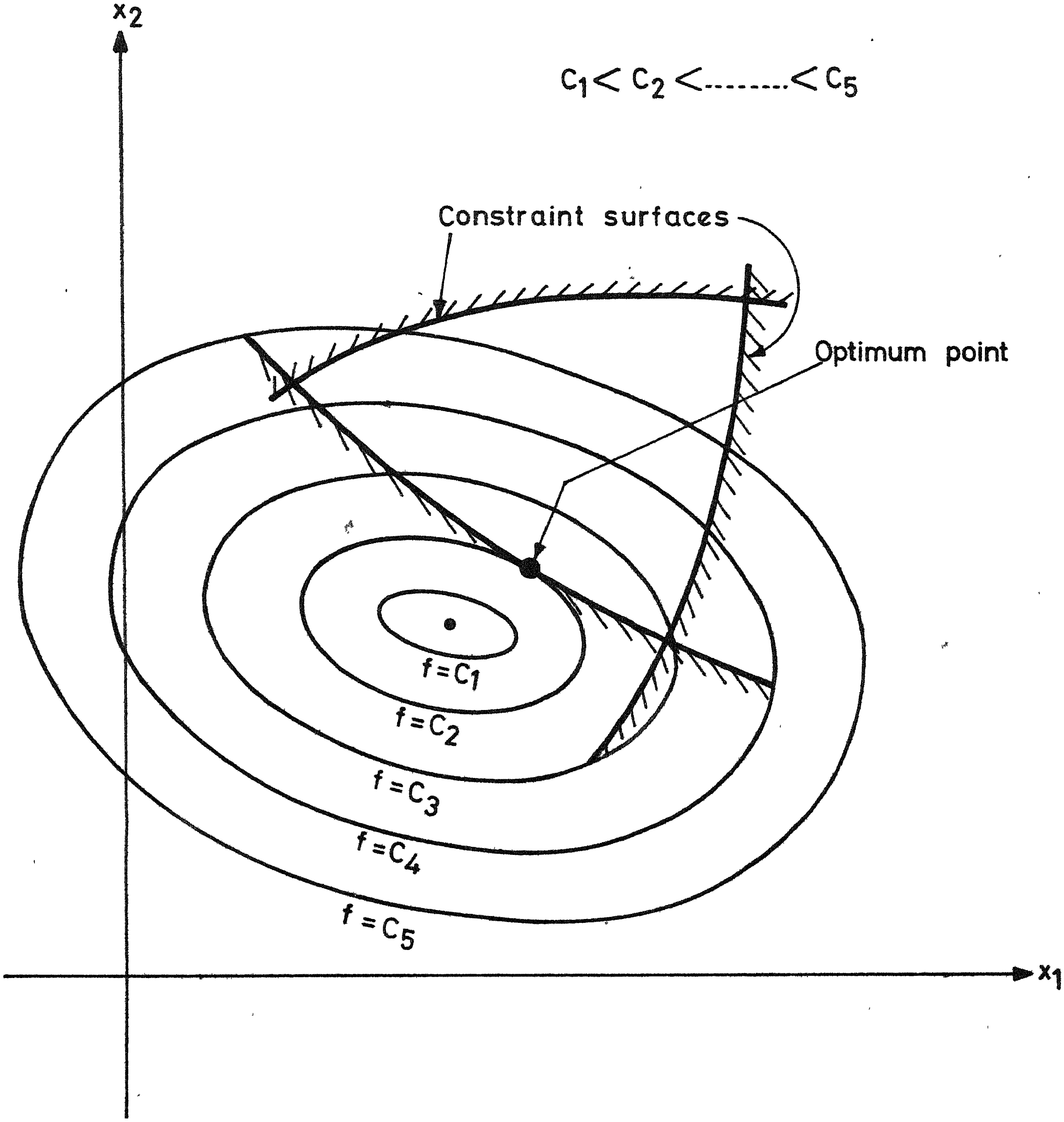


Fig. 1.3 Objective function surfaces.

Once the objective function surfaces are drawn along with the constraint surfaces, the optimum point can be determined without much difficulty. However, as the number of design variables exceeds two or three, the constraint and objective function surfaces become complex even for visualization and the problem has to be solved purely as a mathematical problem.

1.13.6 Gradient

An important concept employed continually in the chapters that follow is that of the *gradient* of the objective function, $\overline{\nabla}f$. The gradient is a vector composed of the derivatives of f with respect to each of the design variables, i.e.

$$\overline{\nabla}f = \begin{Bmatrix} \dfrac{\partial f}{\partial x_1} \\ \dfrac{\partial f}{\partial x_2} \\ \vdots \\ \dfrac{\partial f}{\partial x_n} \end{Bmatrix} \qquad \ldots\ 1.3(a)$$

Similarly the gradient $\overline{\nabla}g_j$ for the jth constraint surface is

$$\overline{\nabla}g_j = \begin{Bmatrix} \dfrac{\partial g_j}{\partial x_1} \\ \dfrac{\partial g_j}{\partial x_2} \\ \vdots \\ \dfrac{\partial g_j}{\partial x_n} \end{Bmatrix} \qquad \ldots\ 1.3(b)$$

1.13.7 Local and Global Minima

A structural designer, while applying the mathematical programming methods, should be able to distinguish between *local* minima and the *global* minimum [Figure 1.4(a)]. If, for example, the optimization procedure leads from the starting point A to the point B, analytical tests applied at point B will indicate that no further moves are possible without a violation of constraints. Thus a local minimum will have been reached. The *absolute* or global minimum, is seen to exist a point C, however.

A local minimum coincides with the global minimum when the constraint surface is *convex*. Mathematically, a function $f(\bar{X})$ is said to be convex for any pair of points

$$\bar{X}_1 = \begin{Bmatrix} x_1^{(1)} \\ x_2^{(1)} \\ \vdots \\ x_n^{(1)} \end{Bmatrix} \quad \text{and} \quad \bar{X}_2 = \begin{Bmatrix} x_1^{(2)} \\ x_2^{(2)} \\ \vdots \\ x_n^{(2)} \end{Bmatrix}$$

and all λ, $0 \leq \lambda \leq 1$,

$$f\,[\lambda.\bar{X}_2 + (1 - \lambda)\bar{X}_1] \leq \lambda f(\bar{X}_2) + (1 - \lambda)\, f(\bar{X}_1) \qquad \ldots\ 1.4$$

Figure 1.4(b) illustrates a convex function in one dimension. It can be easily seen that a convex function is always bending upwards and hence it is apparent that the local minimum of a convex function is also a global minimum.

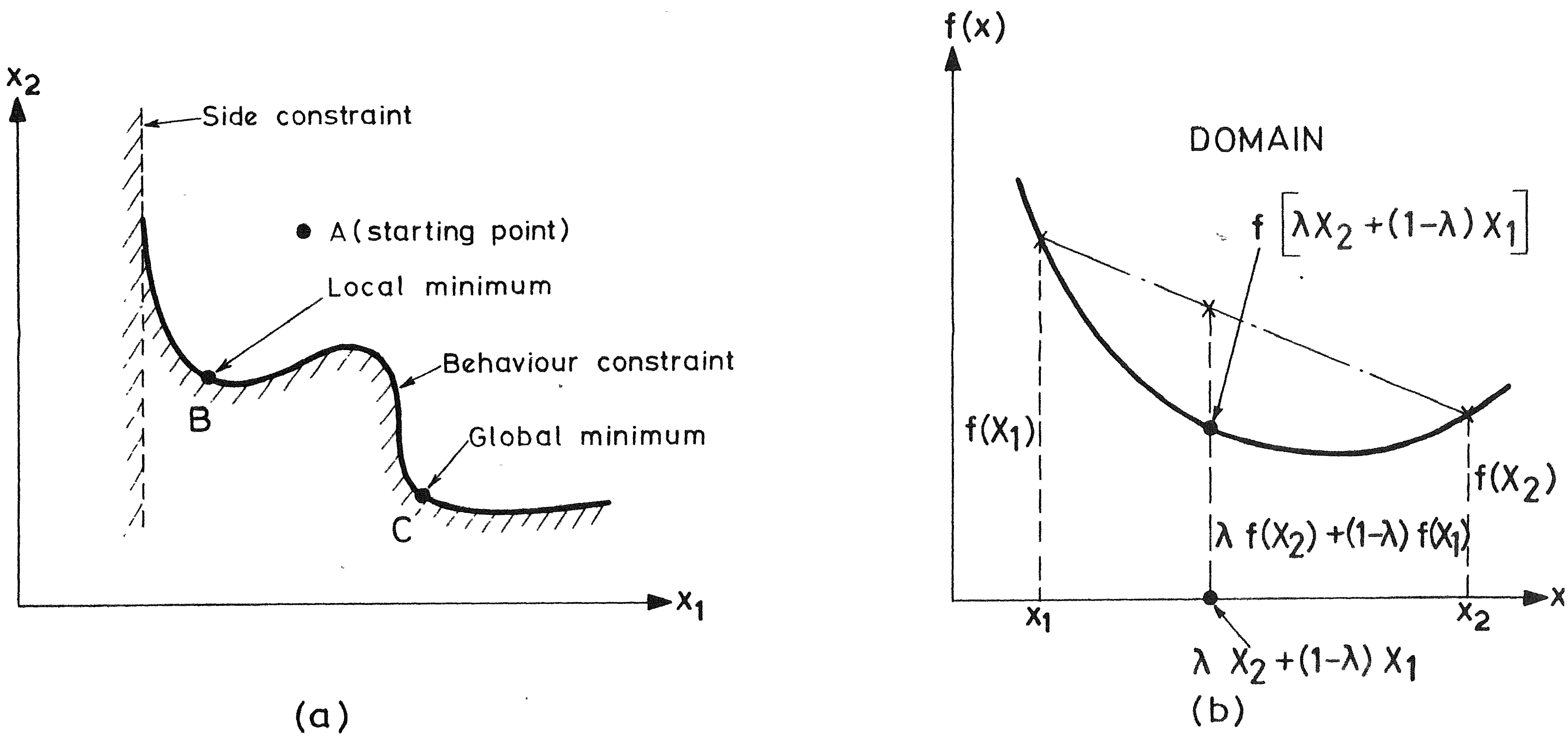


Fig. 1.4 (a) Local and global minima (b) convex function.

Thus, the property of convexity is of value in ascertaining the *uniqueness* of a solution for an optimum design point.

Unfortunately, constraint surfaces in structural design problems are not always convex. Kavlie and Moe (1971), Adidam S.R. (1972), and Moses and Onada (1969) have given examples of structural design problems with concave constraint surfaces.

1.13.8 Kuhn-Tucker Optimality Condition

The analytical test for a local minimum alluded to above is the *Kuhn-Tucker optimality condition* after the men who developed them in 1951. The mathematical statement of this condition is as follows.

In a constrained optimization problem, consider the division of the constraints into two subsets J_1 and J_2 where J_1 indicates the indices of those constraints which are active at the optimum point and J_2 includes the indices of the remaining (inactive) constraints. The Kuhn-Tucker optimality conditions to be satisfied at a constrained minimum point $\bar{X}^*$ can be expressed as

$$\frac{\partial f}{\partial x_i^*} + \sum_{j \in J_1} \lambda_j \frac{\partial g_i}{\partial x_i^*} = 0, \quad i = 1, 2, \ldots, n \qquad \ldots 1.5$$

and $\lambda_j > 0.$

In other words, these are the *necessary* conditions for a local optimum design point, but in general not *sufficient*.

However, there is a class of problems, called *convex programming* problems, wherein both the objective and the constraint functions are convex, the Kuhn-Tucker conditions are necessary and sufficient for a global minimum.

Rewriting Equation 1.5 in the form

$$\bar{\nabla} f(\bar{X}^*) = - \sum_{j \varepsilon J_1} \lambda_j \, \bar{\nabla} g_i(\bar{X}^*) \qquad \ldots 1.6$$

will also mean that in order to satisfy the Kuhn-Tucker optimality conditions, the gradient of the objective function must be expressible as the *negative* of a *linear combination* of the *active constraint gradients*. The non-negative multipliers λ_j $(j \varepsilon J_1)$ are also termed the *Lagrange multipliers* since the development of Equation 1.5 may be accomplished by the application of the Lagrange multiplier approach to the constrained minimization.

1.14 SCOPE OF THE THESIS

The thesis is primarily concerned with the application of a suitable mathematical programming technique to obtain optimal designs of two different classes of civil engineering structures. A reinforced concrete cantilever retaining wall is chosen to represent the thick-walled category. In such structures, the *material cost* being the main objective, optimal designs are obtained for minimum cost for five types of soil commonly met with in practice. As

rolled steel section (I-shape) are quite commonly used in construction, structural I-sections are selected to represent the thin-walled class. In such a case, *weight* being the criterion, optimal designs of beams and columns of I-section are obtained for various types of loading under different sets of constraints viz. the static, the stability and the dynamic.

Both these are essentially constrained optimization problems involving non-linear programming. Two different non-linear programming techniques are employed in the respective problems — in the former, Powell's method is used while in the latter, the solution is attempted using Davidon-Fletcher-Powell's method. In both the problems, the constraints used are those specified by the Indian Codes of Practice.

The thesis is divided into six chapters. Chapter 1 introduces the fundamental concepts of structural analysis and structural design which can be found in any standard textbook on structural theory. The critical aspects that govern categorization of the civil engineering structures into the thick- and thin-walled class are then discussed. A brief historical background of different mathematical programming methods is followed by a selective literature survey of the various non-linear programming techniques. Summarizing, in Chapter 1, it is intended to develop a proper background for the formulation and subsequent solution of the optimization problems alluded before.

In the optimization of the thin-walled class, the important feature is that, the analysis is carried out by the discrete element technique incorporating Vlasov's thin-walled beam theory. The analysis, therefore, considers the effect of warping of the section, an attempt never made so far, for obtaining optimal designs of thin-walled sections. Chapter 2, therefore, discusses the consistent matrix formulation for the discrete element technique for linear and eigenvalue problems of structures assembled from thin-walled segments with open cross-section. The formulation is exact for linear static problems and a close upper bound for eigenvalue problems. The technique presented may be considered as an extension of the procedure used for solid beam structures.

In optimum structural design, the behavioural constraints are usually the prescribed limitations on stresses and deflections. For such a set of *static* constraints, the formulation of the optimization problem for thin-walled sections is discussed in Chapter 3.

Thin-walled component members are widely used in aerospace structures wherein the payload plays a significant part in creating the structural and/or the dynamic instability in the members. In the optimal design of such sections, the behavioural constraints may be in the form of the prescribed buckling strength or the specified range of natural vibrational frequency respectively. Optimal design of thin-walled sections under such sets of the *stability* or the *dynamic* constraints are dealt with in Chapter 4.

An important reason to group these two sets together is that the problem of stability and the problem of free transverse vibrations, both come under the category of the *eigenvalue* problems. Mathematically, an eigenvalue problem in the *general* form consists of determining values of λ for which the 'n' homogeneous *linear* equations in 'n' unknowns

$$[A]\,\bar{X} = \lambda \,.\, [B]\,\bar{X} \qquad \ldots 1.7$$

have a *non-trivial* solution $\bar{X}$ such that

$$\bar{X} = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} \qquad \text{and } x_i \neq 0 \text{ for all i.}$$

The (n x n) matrices [A] and [B] in Equation 1.7 take the form of the *stiffness* and the *geometric* matrices respectively in the stability problem, while in the free transverse vibrations problem, matrix [B] is replaced by the mass matrix.

Chapter 5 discusses the formulation for obtaining optimal designs of cantilever retaining walls for different classes of soil retained. The behavioural constraints take into account different modes of failure of the wall as well as the specifications governing the reinforced concrete design.

In Chapter 6, the results obtained in the preceding chapters have been critically evaluated and conclusions drawn. The thesis concludes with the presentation of scope for further studies.

- o -

2. DISCRETE ELEMENT ANALYSIS OF THIN-WALLED BEAMS

2.1 GENERAL

According to their *spatial* character, structural elements can be classified into four classes viz. the *massive bodies, plates and shells, solid beams* and *thin-walled beams.*

Massive bodies are those whose three dimensions are comparable e.g. a cube, a sphere, an ellipsoid etc. These may be looked upon as an *elastic continuum* filling all space so that the problem of determination of stresses and deformations can be dealt by the *mathematical theory of elasticity.*

Bodies having one dimension (thickness) small compared with the other two (length and breadth) which are of the same order, can be treated as plates and shells. Thin slabs - round, rectangular, trapezoidal etc. in shape belong to the former category while thin shells of various shapes - cylindrical, conical, spherical etc. belong to the latter one. The general theory of such bodies is based on geometrical hypotheses valid for *thin deformable* bodies within a certain degree of approximation.

2.1.1 Solid Beams

Solid beams are bodies characterized such that two of their dimensions, usually the *cross-sectional* ones, are small compared with the third dimension, the *length.* Because of this particular *form* assumed, the analysis of solid beams

involves a number of geometrical hypotheses to simplify the calculations. This may be explained with reference to the following two kinds of basic problems one comes across in the *strength of materials*.

In the problem of transverse bending, out of the six components of the *general strain tensor*, only one corresponding to the direction *parallel to beam axis* is preserved and other five components are *taken* to vanish. Such an assumption is possible because of the *law of plane sections* which means that any cross-section of beam which is *initially* plane, remains plane *after* deformation. This geometrical hypothesis, postulated by Navier, forms the basis of the *elementary theory of bending* of beams.

The problem of torsion follows St. Venant's *theory of pure torsion* i.e. no extension or shearing strain is assumed to exist in the plane of cross-section. In other words, a plane section remains plane i.e. free from any *warpage* or *distortion*, after deformation. This theory allows to determine the tangential stresses in the cross-section of beam.

In general, the class of 'solid beams' includes all beam systems - plane or three-dimensional, statically determinate or indeterminate whose elements are beams following the laws mentioned above.

2.1.2 Thin-Walled Beams

These are bodies characterized by the fact that their *three dimensions* are all of *different* magnitude. They

can be looked upon as essentially having the form of a *long prismatic shell* such that the thickness of cross-section (δ) $\ll$ the characteristic dimension of cross-section (d) $\ll$ the length of the shell (l).

Rolled, welded or rivetted structural steel sections used as columns or beams, elements of girders and frames etc. wherein the ratios δ/d and d/l may be of the *order* of 0.1 or less, can be categorized under thin-walled beams for all practical purposes. Certain engineering structures do have such proportions that they can be considered as thin-walled structures e.g. certain types of girder and arch bridges with sufficiently rigid cross-section, suspension bridges of a roadway with a channel or I-section girder system, long reinforced concrete ribbed structures, cylindrical or prismatic shell arches, bunkers, pipe lines etc.

2.2 DISTINCTIVE FEATURES OF THIN-WALLED BEAMS

When subjected to torsion, thin-walled beams undergo longitudinal extension as a result of *relative* warping of the section, which does not remain *plane*. Hence, *complementary* longitudinal normal stresses proportional to these strains are created which have to maintain the internal equilibrium of longitudinal forces in each cross-section. These are not examined in the theory of pure torsion of solid beams, but can attain large values in thin-walled beams with open (rigid or flexible) cross-section and beams with closed (flexible)

cross-section. Thin-walled beams with closed (rigid) cross-section may be considered as solid beams so far as their behaviour under combined flexure and torsion is considered. Complementary longitudinal stresses that occur in such beams have a *local* character. By St. Venant's principle, they fall off rapidly along the beam.

2.3 VLASOV'S THEORY

In the theory of thin-walled beams of open section, Vlasov (1959) has postulated the following two assumptions:

(a) A thin-walled beam of open section can be considered as a shell of *rigid* (i.e. undeformable) section,

and, (b) Shearing deformation of middle surface of the beam cross-section can be assumed to *vanish*.

These can be so generalized that a loading state at a section can be considered as *superposition* of two *independent* states and the total deformation can be obtained by superposition of deformation in each case.

This can be illustrated referring to Figure 2.1(a) wherein the original loading is viewed as a *statical equivalent* of two independent states shown in Figures 2.1(b) and (c). Obviously, Figure 2.1(b) is a state of *flexure* that can be explained by the *law of plane sections*. Figure 2.1(c) can be looked upon as a typical of *restrained* torsion i.e. torsion in which independent *longitudinal elements* of the beam undergo bending *in addition* to torsion. This is referred to as

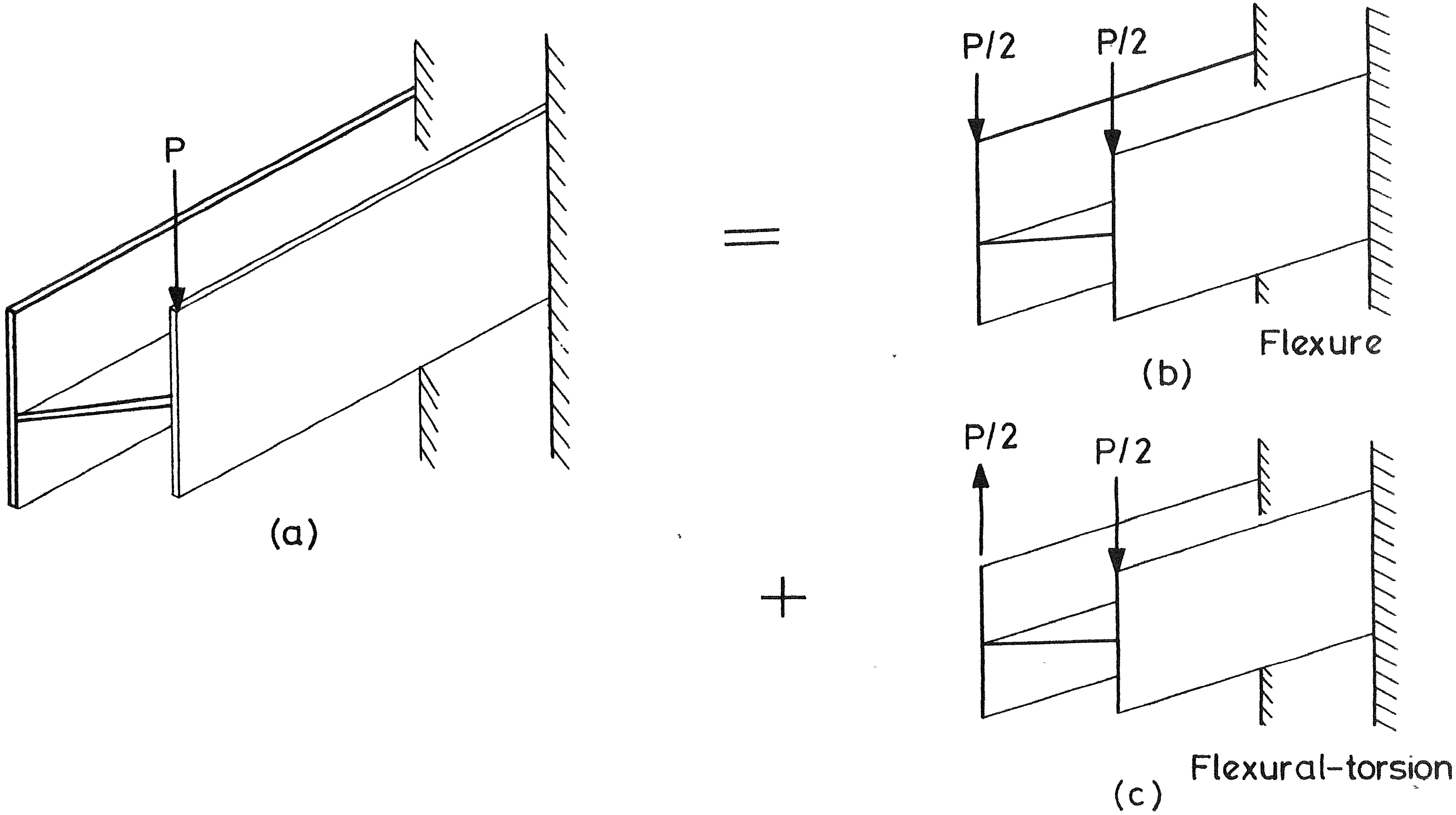


Fig. 2.1 Statical equivalence of loading on a thin-walled beam.

flexural torsion that suggests the existence of *normal stresses* due to flexure in a cross-section in addition to *tangential stresses* due to torsion. As a result, the cross-section does not remain plane - it is warped and hence this kind of torsion is also called the *warping torsion*.

In the case of a thin-walled beam subjected to torsion, the subsequent presence of the longitudinal stresses infers that, part of the work done by the applied twisting moment (T) is used in developing the warping torsion (T_Ω), while the remainder will develop shear stresses associated with the *pure* or the *St. Venant torsion* (T_S). Thus

$$T = T_\Omega + T_S \qquad \ldots 2.1$$

When the warping torsion produces normal stresses at each section of a thin-walled beam, their *stress resultant* is developed as a *generalized force* that has the form of a *system of self-balancing longitudinal tensions*. Such resulting pair of *bending moments* are referred to as *longitudinal bimoment* of the beam. The bimoment has the dimension of force multiplied by area.

2.4 ANALYSIS OF THICK- AND THIN-WALLED BEAMS

To calculate the stresses and strains at a given section of a thick-walled beam, whose behaviour is governed by the *law of plane sections*, one has to know an internal axial force and bending moment together with the area and

moments of inertia of the cross-section. To calculate the deflection of such a beam, one must also know the distribution of external forces along the length of the beam.

All these values are *not sufficient* for the analysis of a thin-walled beam, because the hypothesis of plane sections is no more valid.

Hence Vlasov introduced two *additional, new types* of *forces* termed the *flexural twist* and the *bimoment.* He also introduced *additional* functions of the *sectorial properties* of a section, similar to first and second moments of area. The introduction of these new forces and properties of a section enables a complete analysis for determining the stresses and distortions of thin-walled beams.

2.5 CONSISTENT MATRIX FORMULATION OF THE PROBLEM

In the problem of optimal structural design of rolled steel I-shaped sections, the behavioural constraints have been chosen to be limitations on stresses and deflections *('static' constraints)* as well as prescribed buckling strength *(stability constraints)* and specified range of natural transverse vibrational frequency *(dynamic constraints).* This, being a *multivariate* problem, fits quite well with the recent trend in analysis towards *discrete element* representations. The structure under optimization can be looked upon as an *assemblage* of thin-walled discrete elements with *open* cross-section. Hence, it is essential to *establish* a

consistent matrix formulation for static problems, elastic stability and dynamic response of a structure assembled from thin-walled members of *any open cross-section* in general. An I-shaped section being a particular case of an open section, the foregoing results can be readily used.

Hitherto the theory of flexure and torsion of thin-walled members, having as a distinctive feature the occurrence of normal stresses as a result of torsion (due to warping) has been elaborated. However, as far as the structural systems assembled from such members are concerned, very little has been done to treat the problem on the lines of modern techniques. Since the single thin-walled member is by itself statically indeterminate regardless of boundary conditions, the number of redundant forces is considerably higher than for a similar structure assembled from solid beams. This undoubtedly emphasises the need for the development of a method suitable for the application of *high speed digital computers*.

Krahula (1967) derived the stiffness matrix for a thin-walled member. One has to mention also the paper by Renton (1967) considering one of the aspects of stability. Both these papers however lacked *generality* and *consistency* of corresponding procedures developed for solid beam structures by Archer (1963, 1965) and Hartz (1965). Kollbrunner (1968) developed a displacement method and a new calculation model for the thin-walled bars. Krajcinovic (1969) developed a consistent and general matrix formulation of thin-walled beams

applicable to all *linear* and *linearized* structural problems. Recently, Nitssche (1976) developed stiffness matrices for thin-walled beam members of closed section subjected to flexure and torsion, considering the effects of warping torsion.

2.6 FORMULATION OF THE PROBLEM

A thin-walled member, as a *basic* element of a structure articulated from many such components, is considered as a *spatial* system composed of plates undergoing both bending and axial stressing in plane. The general theory of thin-walled members of open cross-section, as developed in its final form by Vlasov (1959), is essentially used in the formulation. The two basic assumptions on which the theory is based are enumerated in Article 2.3 and these reduce the number of degrees of freedom at each cross-section to *four*. Thin-walled beams treated according to such a theory are distinguished from solid beams by experiencing longitudinal strains as a result of torsion (due to warping). In other words, instead of Navier's hypothesis of plane sections, a more *general* hypothesis governing the *kinematics* of the member is introduced.

Consider a member having a thin-walled, open cross-section, as shown in Figure 2.2. The section being rigid, the displacement of a point on the middle surface of the member can be described by three component displacements ξ, η and w_o in the direction of the principal axes x, y and z

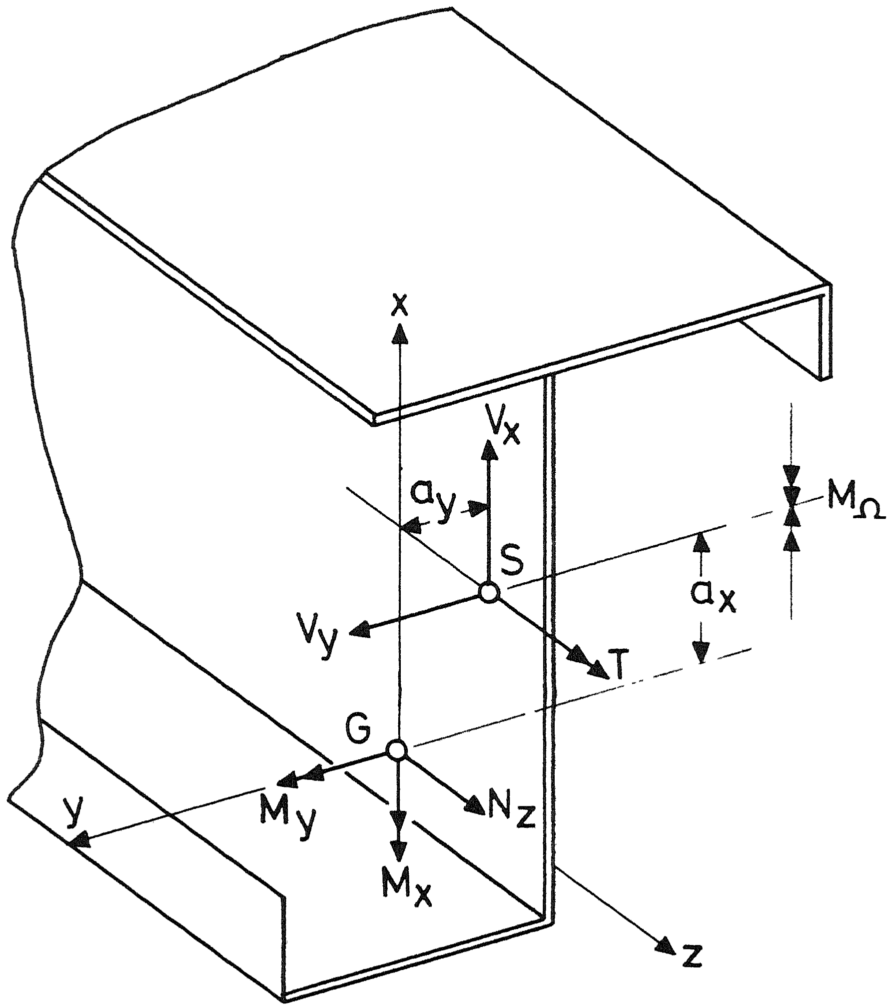


Fig. 2.2 Thin-walled open cross-section.

where d_i and δ_i are the breadth and thickness of the i^{th} plate; $I_{\Omega\Omega}$ is the sectorial moment of inertia defined by

$$I_{\Omega\Omega} = \int \Omega^2 \, dA \qquad \ldots \; 2.3(c)$$

where Ω is the *normalized* sectional co-ordinate; and

$$T = T_\Omega + T_s \qquad \ldots \; 2.1$$

The total torque 'T' is a *vector* sum of the St. Venant's (T_s) and the warping torque (T_Ω) as already indicated in Equation 2.1. The primes denote differentiation with respect to z.

It should be noted that Equations 2.2 imply the *orthogonality* of co-ordinates x, y and Ω or that in other words, the normal force N_z and bending moments M_x and M_y are reduced to the centre of gravity G, while the torque T and transverse forces V_x and V_y are reduced at the shear centre S (Figure 2.2). It may also be noted that the transverse forces V_x and V_y cannot be defined in terms of deformations as a result of imposed assumptions about the deformation.

The equations of equilibrium in terms of componental deformations for the linear static problem are due to the orthogonality of x, y and Ω *uncoupled.*

z-direction, $$EAw_o'' = -p_z \qquad \ldots \; 2.4(a)$$

x-direction, $$EI_{xx}\xi'''' = p_x + m_x' \qquad \ldots \; 2.4(b)$$

y-direction, $$EI_{yy}\eta'''' = p_y + m_y' \qquad \ldots \; 2.4(c)$$

Moment equilibrium about z-direction, $EI_{\Omega\Omega}\varphi'''' - GK\varphi'' = m_t + m_\Omega'$ ... 2.4(d)

where p_x, p_y and p_z are distributed loads *in the directions* of the x, y and z axes respectively while m_x, m_y, m_t and m_Ω are distributed bending moments, torque and bimoment respectively acting upon the member.

Equation 2.4(a) governs the problem of an *axially stressed member* while Equations 2.4(b) and (c) govern the problem of *flexure in the two principal planes*. Equation 2.4(d) governs the problem of *warping torsion* of a thin-walled member.

Considering the *order* of differential equations in relations 2.4(a) through (d), a *set* of fourteen boundary conditions (seven for each terminal cross-section of the member) should be attached to this system in order to define the ***boundary** value problem*. Such seven possible displacements, called the *nodal degrees of freedom* for *each terminal* cross-section of the element are: ξ, η, w_o, the *displacements* along axes x, y and z; ξ', η' and φ, the *rotations* about these axes, and φ', the *warping* of the section. The solution of the problem is attempted by *assuming* a *displacement mode*.

2.7 DISPLACEMENT FUNCTIONS

Define the displacement field q(z, t) as

$$q(z,t) = \sum_i q_i(t)\,\gamma_i(z)$$

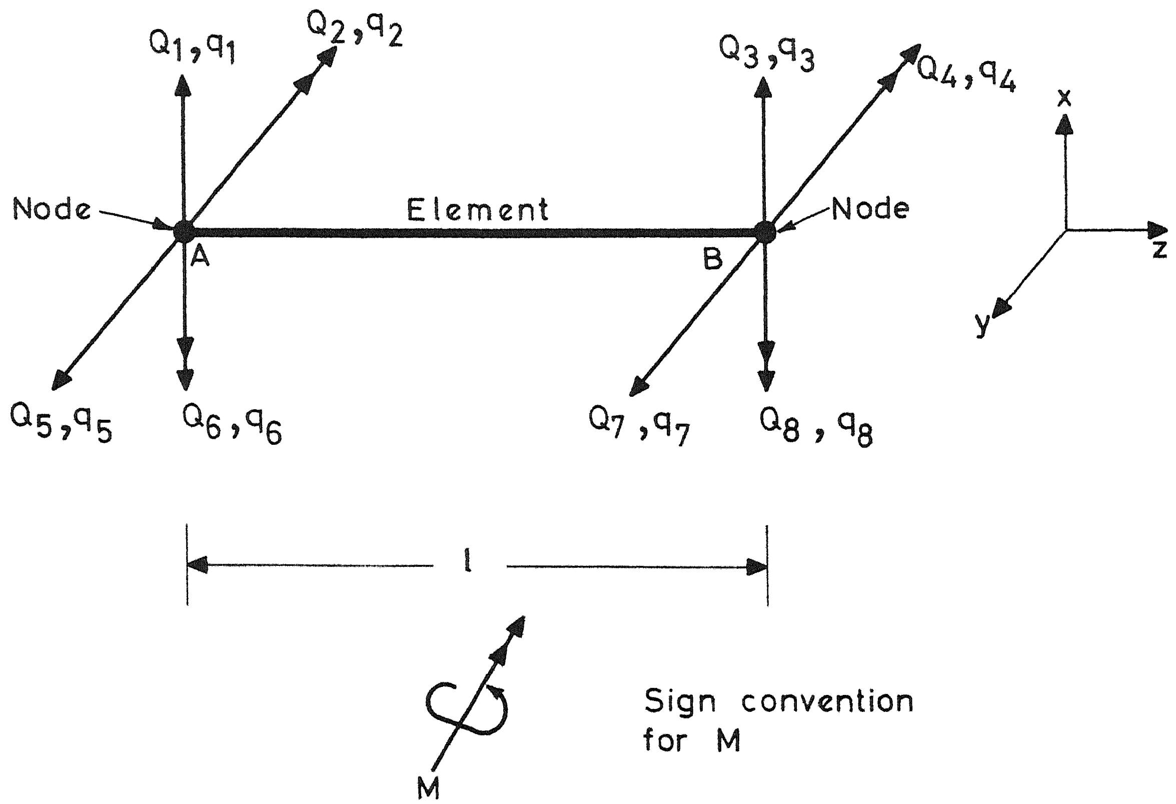
$$= q_1\gamma_1 + q_2\gamma_2 + \dots + q_n\gamma_n \qquad \dots\ 2.5$$

where $\gamma_i(z)$ are some *assumed* displacement modes; q(t) are the *unknown* amplitudes at the structural *nodes* i.e. the *nodal displacements* and n is the total number of generalized nodal displacements, so chosen that it *equals* the nodal degrees of freedom for an element.

Moreover, the vector $\{q_i\} = \begin{Bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{Bmatrix}$ can be subdivided into four subvectors

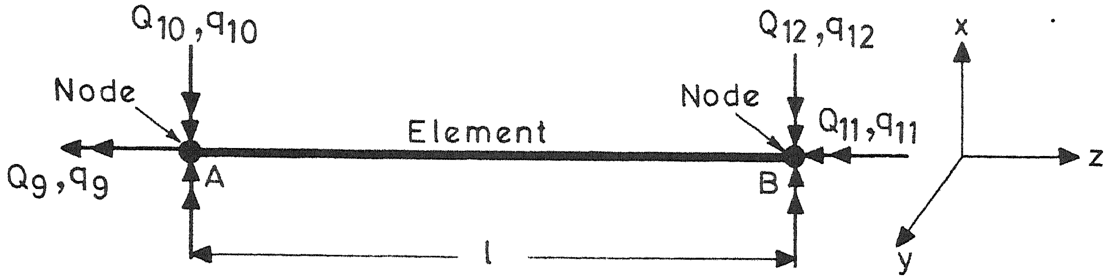
$$\{q_i\} = \begin{Bmatrix} q^z \\ q^x \\ q^y \\ q^\varphi \end{Bmatrix} \qquad \ldots \; 2.6$$

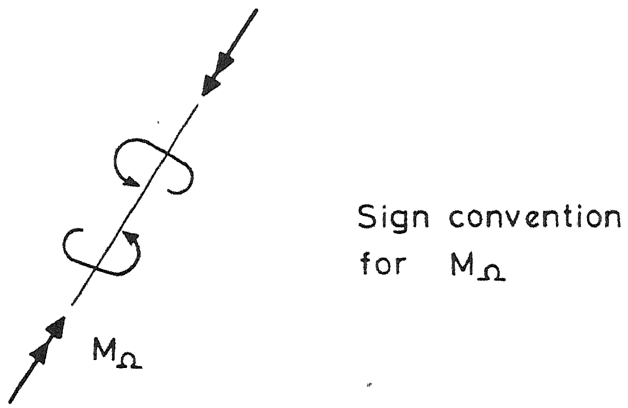
The first two generalized displacements (subvector q^z), characterize the *axial stressing* of the member. These, in sequel, be neglected as the axial deformations are expected to be of the *small order*. Modes $\gamma_i(z)$ corresponding to next eight nodal displacements q_i (subvectors q^x and q^y) characterize the *flexure* of beam in the two principal planes, while the remaining four q_i (subvector q^φ) characterize its *flexural torsion*. The corresponding *twelve* nodal displacements characterizing the *twelve* displacement modes, are shown in Figures 2.3 and 2.4.



Q_i	Forces (kg)	q_i	Displacement (cm)
Q_1 , Q_3	V_x	q_1 , q_3	ξ
Q_2 , Q_4	M_y / l	q_2, q_4	$\xi' l$
Q_5, Q_7	V_y	q_5, q_7	η
Q_6, Q_8	M_x / l	q_6, q_8	$\eta' l$

Fig. 2.3 Flexure problem





Q_i	Forces (kg)	q_i	Displacement (cm)
Q_9 , Q_{11}	T/l	q_9 , q_{11}	ψl
Q_{10} , Q_{12}	M_Ω/l^2	q_{10} , q_{12}	$\psi' l^2$

Fig. 2.4 Flexural torsion problem.

2.7.1 Flexure

For flexure in the principal plane x-z, the first four displacement functions can be obtained with reference to Figure 2.5. The displacement mode $\gamma_1(z)$ is the equation of the *elastic curve* corresponding to $q_1 = 1$ and all *other* displacements remaining zero [Figure 2.5(a)]. The governing Bernoulli-Euler equation for flexure is given by

$$EI_{xx} \frac{d\xi^4}{dz^4} = 0 \qquad \ldots\ 2.7$$

Successive integration leads to

$$\xi = c_1 \frac{z^3}{6} + c_2 \frac{z^2}{2} + c_3 z + c_4 \qquad \ldots\ 2.8$$

where c_1, c_2, c_3 and c_4 are the constants of integration.

Using the boundary conditions

$$\xi = 1, \quad \xi' l = 0, \quad \text{at} \quad z = 0$$

$$\text{and} \quad \xi = \xi' l = 0, \quad \text{at} \quad z = l \qquad \ldots\ 2.9$$

the displacement function for the ***first*** mode is obtained as

$$\gamma_1(z) = \left(1 + \frac{2z}{l}\right)\left(1 - \frac{z}{l}\right)^2 \qquad \ldots\ 2.10(a)$$

Similarly the *second, third* and *fourth* modes in flexure are

$$\gamma_2(z) = -\frac{z}{l}\left(1 - \frac{z}{l}\right)^2, \qquad \ldots\ 2.10(b)$$

$$\gamma_3(z) = \frac{z^2}{l^2}\left(3 - \frac{2z}{l}\right), \qquad \ldots\ 2.10(c)$$

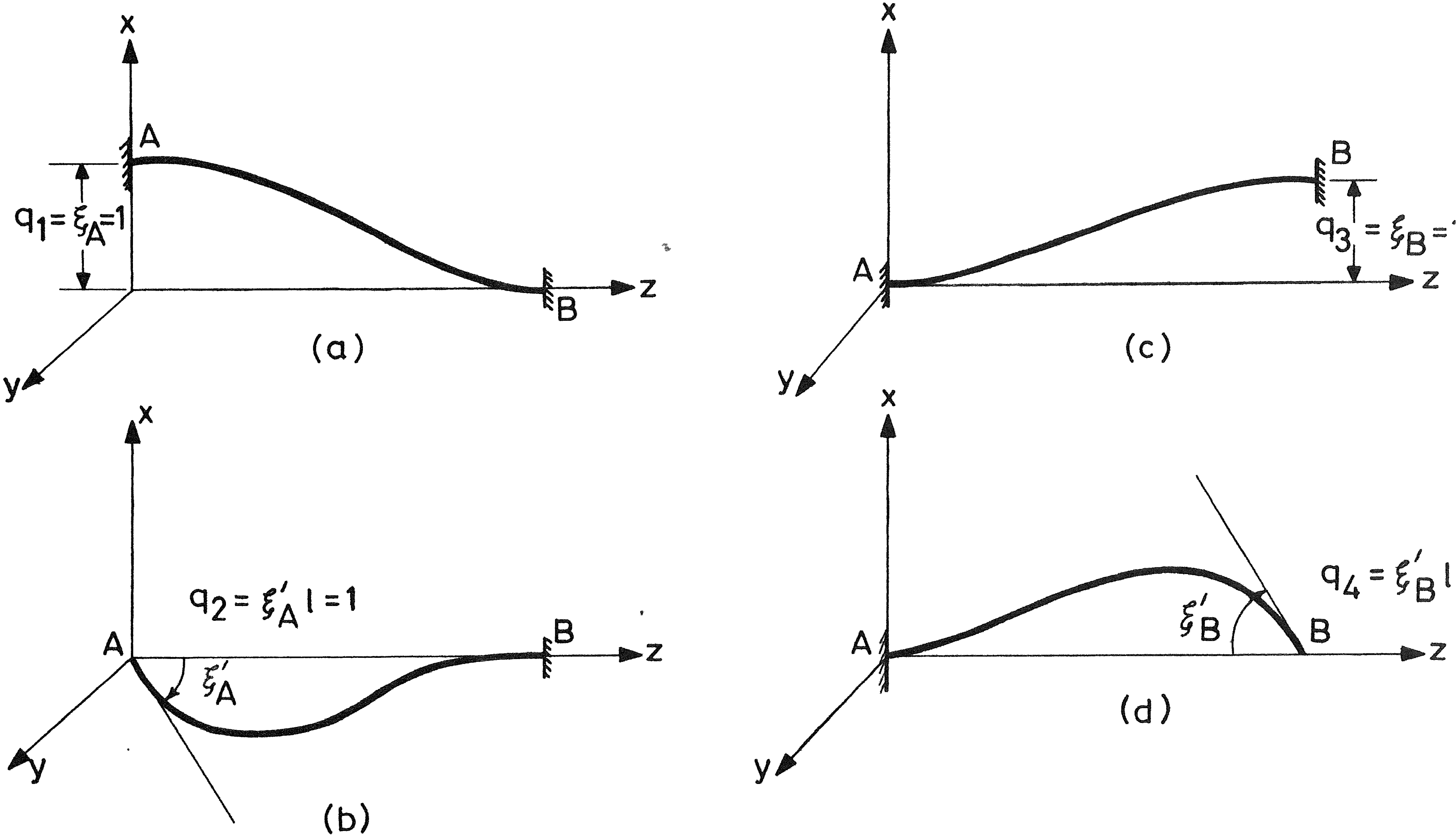


Fig. 2.5 Displacement functions for flexure.

and $$\gamma_4(z) = \frac{z^2}{l^2}(1 - \frac{z}{l}) . \qquad \text{... 2.10(d)}$$

The foregoing γ's are the set of displacement modes for flexure in the principal plane x-z.

Proceeding on similar lines, corresponding set of displacement modes for flexure in the principal plane y-z is

$$\gamma_5(z) = (1 + \frac{2z}{l})(1 - \frac{z}{l})^2 ,$$

$$\gamma_6(z) = \frac{z}{l}(1 - \frac{z}{l})^2 ,$$

$$\gamma_7(z) = \frac{z^2}{l^2}(3 - \frac{2z}{l}) , \qquad \text{... 2.11}$$

and $$\gamma_8(z) = -\frac{z^2}{l^2}(1 - \frac{z}{l}) .$$

2.7.2 Flexural Torsion

Starting from Equation 2.4(d), the solution of the homogeneous part is

$$\varphi(z, t) = c_5 \text{ch}kz + c_6 \text{sh}kz + c_7 kz + c_8 \qquad \text{... 2.12}$$

where $$k^2 = GK/EI_{\Omega\Omega} \qquad \text{... 2.13}$$

Again, γ_i is the solution to the homogeneous equation for $q_i = 1$ and all other $q_j = 0$ ($i = 9, \ldots, 12$; $j = 1, 2, \ldots, 12$; $i \neq j$), so that

$$\gamma_9(z) \quad = \frac{1}{lD}\,[(1 - \text{ch}\,\kappa)\,\text{ch}\,kz + \text{sh}\,\kappa\,\text{sh}\,kz$$

$$- kz\,\text{sh}\,\kappa + 1 - \text{ch}\,\kappa + \kappa\,\text{sh}\,\kappa]$$

$$\gamma_{10}(z) \quad = \frac{1}{\kappa l^2 D}\,[(\kappa\,\text{ch}\,\kappa - \text{sh}\,\kappa)\,\text{ch}\,kz + (\text{ch}\,\kappa - 1$$

$$- \kappa\,\text{sh}\,\kappa)\,\text{sh}\,kz + kz(\text{ch}\,\kappa - 1) + \text{sh}\,\kappa - \kappa\,\text{ch}\,\kappa] \qquad \ldots\ 2.14$$

$$\gamma_{11}(z) \quad = \frac{1}{lD}\,[(\text{ch}\,\kappa - 1)\,\text{ch}\,kz - \text{sh}\,\kappa\,\text{sh}\,kz$$

$$+ kz\,\text{sh}\,\kappa - 1 - \text{ch}\,\kappa]$$

and

$$\gamma_{12}(z) \quad = \frac{1}{\kappa l^2 D}\,[(\text{sh}\,\kappa - \kappa)\,\text{ch}\,kz + (1 - \text{ch}\,\kappa)\,\text{sh}\,kz$$

$$+ kz + kz(\text{ch}\,\kappa - 1) + \kappa - \text{sh}\,\kappa]$$

where

$$D \quad = \quad 2(1 - \text{ch}\,\kappa) + \kappa\,\text{sh}\,\kappa \qquad \ldots\ 2.15$$

and $\kappa \quad = \quad kl$ $\qquad \ldots\ 2.16$

with 'l' denoting the length of the member. The symbols 'ch' and 'sh' refer to the hyperbolic cosine and sine functions respectively.

For the generalized nodal displacements and the corresponding generalized nodal forces shown in Figures 2.3 and 2.4, for the element under consideration,

the nodal displacement vector $\{q_i\} = [q_1, q_2, \ldots, q_{12}]^T$

$$= [\xi_A, \xi_A' l, \xi_B, \xi_B' l, \eta_A, \eta_A' l, \eta_B, \eta_B' l, \varphi_A l, \varphi_A' l^2, \varphi_B l, \varphi_B' l^2]^T \qquad \ldots 2.17$$

and the generalized nodal force vector $\{Q_i\} = [Q_1, Q_2, \ldots, Q_{12}]^T$

$$= [V_{xA}, M_{yA}/l, V_{xB}, M_{yB}/l, V_{yA}, M_{xA}/l, V_{yB}, M_{xB}/l, T_A/l, M_{\Omega A}/l^2, T_B/l, M_{\Omega B}/l^2]^T . \qquad \ldots 2.18$$

The potential energy may then be written as a scalar product

$$V = \{q_i\}^T \{Q_i\} \qquad \ldots 2.19$$

Some displacements and forces are multiplied by the length of the beam l to various powers in order to achieve dimensional homogeneity. The nodal displacements q_i have the dimension of length while the generalized nodal forces Q_i have the dimension of a force.

2.8 NODAL FORCE VECTOR

The nodal force vector $\{Q_i\}$ has to be determined at a number of nodes *(structural co-ordinates)* that characterize the division of the structure *(discretization)* into a finite number of elements, and then it is *assembled* for the entire structure. The load applied to the structure can, in general, be either distributed or concentrated at certain locations

(not necessarily coinciding with the nodes) or both. It is therefore necessary to establish a relation between the nodal force vector $\{Q_i\}$ and *external* loads, which will be *consistent* with the *structural idealization*. In order to achieve this, one can obtain the *equivalent nodal loads* vector $\{F_i\}$ by considering the strain energy equivalence of the deformed structure with the original external loads. The nodal force vector $\{Q_i\}$ can then be determined as the *statical equivalent* to the vector $\{F_i\}$.

For a flexure problem, one can obtain the nodal loads vector $\{F_i\}$ for any *type of loading* by using necessary relations given in Martin (1966) or Gere (1969) and thus can obtain $\{Q_i\}$.

Recalling the basic principles of the theory of structures and in particular the Mueller-Breslau theorem, one can define the displacement modes as the *influence functions* for the torsional modes (γ_9 and γ_{11}) and bimoment (γ_{10} and γ_{12}) for a beam with both ends fully clamped (in sense of *vanishing deplanation* and *twist*). Having recognized this, one can write the relation for the generalized nodal force vector in the case of flexural torsion problems as

$$\{Q_i\} = \int_o^l \gamma_i(z)\ m_t(z)\ dz - \int_o^l \gamma_i'(z)\ m_\Omega(z)\ dz \qquad \ldots\ 2.20$$

In case of concentrated forces, one can use Dirac delta functions to evaluate integrals in Equation 2.20.

The same relation has been obtained by Archer (1965) using virtual work argument and Betti's principle.

2.9 MATRIX FORMULATION OF THE PROBLEM

The general *non-linear* dynamic problem of elastic thin-walled structures presents considerable mathematical difficulties. However, a majority of problems of practical significance may be treated in a much simpler form assuming the *steady state motion* of the structure and *constancy* of parametric loads. This *linearization* essentially transforms the non-linear problem into an *eigenvalue problem* of much simpler form.

In accord with the adopted model, it is further assumed that the loads are reduced to nodes. Although this assumption is not *central* to the derivation, it results in a much simpler form for all the governing matrices.

The *linearized* differential equations of *dynamic equilibrium* defining the eigenvalue problem are (Figure 2.2):

Transverse vibrations in x-z plane,

$$EI_{xx}\xi'''' + P\,\xi'' + (M_x + a_y P)\,\varphi'' + \rho A(\ddot{\xi} + a_y\,\ddot{\varphi}) = 0 \quad \ldots\ 2.21(a)$$

Transverse vibrations in y-z plane,

$$EI_{yy}\eta'''' + P\,\eta'' + (M_y - a_x P)\,\varphi'' + \rho A(\ddot{\eta} - a_x\,\ddot{\varphi}) = 0 \quad \ldots\ 2.21(b)$$

Torsional vibrations about the longitudinal axis through the shear centre,

$$EI_{\Omega\Omega}\,\varphi'''' + (r^2P + 2\beta_x M_y - 2\beta_y M_x - GK)\,\varphi'' + (M_x + a_y P)\,\xi''$$

$$+ (M_y - a_x P)\,\eta'' + \rho A(a_y\ddot{\xi} - a_x\ddot{\eta} + r^2\ddot{\varphi}) = 0 \qquad \ldots\ 2.21(c)$$

where terms in addition to those in Equations 2.4 are either due to the *distortion* of the beam or to the *inertia*. External load is expressed through the resultant axial force P and resultant bending couples M_x and M_y. The co-ordinates of the shear centre are a_x and a_y, and ρ is the mass per unit volume of the beam material. Other parameters are

$$r^2 = \frac{I_{xx} + I_{yy}}{A} + a_x^2 + a_y^2 \qquad \ldots\ 2.22(a)$$

$$\beta_x = \frac{1}{2I_{xx}} \int x(x^2 + y^2)\,dA - a_x \qquad \ldots\ 2.22(b)$$

and $$\beta_y = \frac{1}{2I_{yy}} \int y(x^2 + y^2)\,dA - a_y . \qquad \ldots\ 2.22(c)$$

The differentiation with respect to time is indicated by dots.

It may be noted that the left hand sides of Equations 2.21 represent the resultant forces in direction of co-ordinate axes x and y and the resulting torque about the shear centre axis. Multiplying them by a set of virtual displacements $\delta\xi$, $\delta\eta$ and $\delta\varphi$ and integrating over the whole *domain* one gets from Equations 2.21

$$\int_0^1 \{[EI_{xx}\xi'''' + P\xi'' + (M_x + a_yP)\varphi'' + \rho A(\ddot{\xi} + a_y\ddot{\varphi})]\delta\xi$$

$$+ [EI_{yy}\eta'''' + P\eta'' + (M_y - a_xP)\varphi'' + \rho A(\ddot{\eta} - a_x\ddot{\varphi})]\delta\eta$$

$$+ [EI_{\Omega\Omega}\varphi'''' + (r^2P + 2\beta_xM_y - 2\beta_yM_x - GK)\varphi'' + (M_x + a_yP)\xi''$$

$$+ (M_y - a_xP)\eta'' + \rho A(a_y\ddot{\xi} - a_x\ddot{\eta} + r^2\ddot{\varphi})]\delta\varphi\}\, dz = 0$$

... 2.23

Equation 2.23 simply states that the *work of all internal and external forces of an equilibrated structure vanishes for any arbitrary system of kinematically admissible virtual displacements*, which is the well-known ***principle of virtual work.***

It may be recognized that Equation 2.23 may be interpreted as the *Galerkin formulation.*

Integration of Equation 2.23 by parts, with the interchange of derivation and variation leads to

$$\int_0^1 [EI_{xx}\xi''\delta(\xi'') + EI_{yy}\eta''\delta(\eta'') + EI_{\Omega\Omega}\varphi''\delta(\varphi'') + GK\varphi'\delta(\varphi')]\, dz$$

$$+ \int_0^1 \{P[\xi'\delta(\xi') + \eta'\delta(\eta')] + 2(M_x + a_yP)\varphi'\delta(\xi')$$

$$+ 2(M_y - a_xP)\varphi'\delta(\eta') + (r^2P + 2\beta_xM_y - 2\beta_yM_x)\varphi'\delta(\varphi')\}\, dz$$

$$+ \rho\int_0^1 \{A[\ddot{\xi}\delta(\xi) + \ddot{\eta}\delta(\eta) + r^2\ddot{\varphi}\delta(\varphi)] + I_{xx}\ddot{\xi}'\delta(\xi')$$

$$+ I_{yy}\ddot{\eta}'\delta(\eta') + I_{\Omega\Omega}\ddot{\varphi}'\delta(\varphi') + 2A[a_y\ddot{\varphi}\delta(\varphi) - a_x\ddot{\varphi}\delta(\eta)]\}\, dz$$

$$= J_0(0, 1)$$

... 2.24

where $J_o(0, 1)$, usually called *conjunct* or *concomitant* is the sum of all integrated terms. For *natural boundary conditions*, to be treated in sequel, conjunct vanishes and the problem is said to be *self-adjoint*.

Introducing for displacements $q(z, t)$, Equation 2.5, Equation 2.24 may be written in the form

$$\{\delta q_i\} ([k_{ij}] - [\tilde{g}_{ij}]) \{q_j\} + \{\delta q_i\}[m_{ij}]\{\ddot{q}_j\} = 0 \quad \dots 2.25$$

where $[k_{ij}]$, $[g_{ij}]$ and $[m_{ij}]$ are called the *stiffness*, the *stability* and the *mass matrix* respectively. $[\tilde{g}_{ij}]$ denotes the form of $[g_{ij}]$, multiplied by a *scalar*.

Finally, making use of *Castigliano's first theorem*, one obtains

$$([k_{ij}] - [g_{ij}]) \{q_j\} + [m_{ij}]\{\ddot{q}_j\} = \{Q_i\} \quad \dots 2.26$$

where $\{Q_i\}$ is the vector of generalized nodal forces defined previously in Equation 2.18.

It may be noted that the most frequently used way of derivation starting from stresses and strains is not a straightforward one in this case. This is due to the basic assumption about the deformation.

2.10 STIFFNESS, STABILITY AND MASS MATRICES

2.10.1 Stiffness Matrix

The stiffness matrix $[k_{ij}]$ is defined by the *first* integral on the left hand side of Equation 2.24. It is

apparently a quasidiagonal matrix of twelth order, of the form

$$[k_{ij}] = \begin{bmatrix} k^{xx} & 0 & 0 \\ 0 & k^{yy} & 0 \\ 0 & 0 & k^{\varphi\varphi} \end{bmatrix} \quad \ldots\ 2.27$$

The first two submatrices representing stiffnesses in two principal planes are well-known from standard references such as Przemieniecki (1968), Zienkiewicz (1971) etc. When expressed in the non-dimensional form, these are

$$[k^{xx}] = \frac{EI_{xx}}{l^3} \begin{bmatrix} 12 & -6 & -12 & -6 \\ -6 & 4 & 6 & 2 \\ -12 & 6 & 12 & 6 \\ -6 & 2 & 6 & 4 \end{bmatrix} \quad \ldots\ 2.28(a)$$

and $$[k^{yy}] = \frac{EI_{yy}}{l^3} \begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix} . \quad \ldots\ 2.28(b)$$

The elements of matrix $[k^{\varphi\varphi}]$ are

$$k_{ij}^{\varphi\varphi} = EI_{\Omega\Omega} \int_0^1 (\Upsilon_i'' \Upsilon_j'' + k^2\, \Upsilon_i' \Upsilon_j')\, dz \quad \ldots\ 2.29(a)$$

$$i, j = 9, \ldots, 12$$

so that the final elemental submatrix $[k^{\varphi\varphi}]$ reads

$$[k^{\varphi\varphi}] = \frac{EI_{\Omega\Omega}}{Dl^5}\begin{bmatrix} \kappa^3 sh\kappa & \kappa^2(1-ch\ \kappa) & -\kappa^3 sh\kappa & \kappa^2(1-ch\ \kappa) \\ & \kappa(\kappa\ ch\kappa - sh\ \kappa) & \kappa^2(ch\kappa - 1) & \kappa(sh\kappa - \kappa) \\ & & \kappa^3 sh\kappa & \kappa^2(ch\kappa - 1) \\ \text{Symm.} & & & \kappa(\kappa\ ch\kappa - sh\kappa) \end{bmatrix} \quad \ldots\ 2.29$$

where D and κ are given by relations 2.15 and 2.16 respectively.

2.10.2 Stability Matrix

The elemental stability matrix $[g_{ij}]$ as defined by the *second* integral in Equation 2.24 is of the form

$$[g_{ij}] = \begin{bmatrix} g^{xx} & 0 & g^{x\varphi} \\ 0 & g^{yy} & g^{y\varphi} \\ g^{\varphi x} & g^{\varphi y} & g^{\varphi\varphi} \end{bmatrix} \quad \ldots\ 2.30$$

Again, referring to Przemieniecki (1969), submatrices $[g^{xx}]$ and $[g^{yy}]$, corresponding to flexure in the two principal planes, can be represented in the non-dimensional form as

$$[g^{xx}] = -\frac{P}{l}\begin{bmatrix} \frac{6}{5} & -\frac{1}{10} & -\frac{6}{5} & -\frac{1}{10} \\ -\frac{1}{10} & \frac{2}{15} & \frac{1}{10} & -\frac{1}{30} \\ -\frac{6}{5} & \frac{1}{10} & \frac{6}{5} & \frac{1}{10} \\ -\frac{1}{10} & -\frac{1}{30} & \frac{1}{10} & \frac{2}{15} \end{bmatrix} \quad \ldots\ 2.31(a)$$

and

$$[g^{yy}] = -\frac{P}{l}\begin{bmatrix} \frac{6}{5} & \frac{1}{10} & -\frac{6}{5} & \frac{1}{10} \\ \frac{1}{10} & \frac{2}{15} & -\frac{1}{10} & -\frac{1}{30} \\ -\frac{6}{5} & -\frac{1}{10} & +\frac{6}{5} & -\frac{1}{10} \\ \frac{1}{10} & -\frac{1}{30} & -\frac{1}{10} & \frac{2}{15} \end{bmatrix} \quad \ldots\ 2.31(b)$$

the axial load P being considered as *compressive*. As the submatrices $[g^{\varphi x}]$ and $[g^{\varphi y}]$ are transposes of $[g^{x\varphi}]$ and $[g^{y\varphi}]$ respectively, using the second integral of Equation 2.24 one needs to evaluate submatrices $[g^{x\varphi}]$, $[g^{y\varphi}]$ and $[g^{\varphi\varphi}]$ only. The elements of these *symmetric* matrices are:

$$g^{x\varphi}_{ij} = 2\int_0^l (a_y P + M_x)\, \gamma_i'\gamma_j'\, dz \quad \ldots\ 2.32(a)$$

$(i = 1, \ldots, 4, \quad j = 9, \ldots, 12)$

$$g^{y\varphi}_{ij} = -2\int_0^l (a_x P - M_y)\, \gamma_i'\gamma_j'\, dz \quad \ldots\ 2.33$$

$(i = 5, \ldots, 8, \quad j = 9, \ldots, 12)$

and $$g^{\varphi\varphi} = \int_0^l (r^2 P + 2\beta_x M_y - 2\beta_y M_x)\, \gamma_i'\gamma_j'\, dz \quad \ldots\ 2.34(a)$$

$(i, j = 9, \ldots, 12)$.

2.10.2.1 Submatrices $[g^{x\varphi}]$ and $[g^{y\varphi}]$

Making use of Equation 2.32(a), it is sufficient to generate the following coefficients

$$g_{1,9} = -g_{3,9} = -g_{1,11} = g_{3,11}$$

$$= -\frac{12}{l^2\kappa^2}\left[1 - \frac{\kappa^3}{12(\kappa - 2\,\mathrm{th}\,\frac{1}{2}\kappa)}\right]$$

$$g_{2,9} = g_{4,9} = -g_{2,11} = -g_{4,11} = g_{1,10} = -g_{3,10}$$

$$= g_{1,12} = -g_{3,12} \doteq \frac{6}{l^2\kappa^2}\left[1 + \frac{\kappa^2}{12} - \frac{\kappa^3}{12(\kappa - 2\mathrm{th}\,\frac{1}{2}\kappa)}\right]$$

$$g_{2,10} = g_{4,12} = \frac{4}{l^2\kappa^2}\left[-1 + \frac{\kappa}{4D}(\kappa\,\mathrm{ch}\,\kappa - \mathrm{sh}\,\kappa)\right]$$

and $$g_{4,10} = g_{2,12} = \frac{2}{l^2\kappa^2}\left[-1 - \frac{\kappa}{2D}(\kappa - \mathrm{sh}\,\kappa)\right]$$

... 2.32(b)

where the symbol 'th' refers to the hyperbolic tangent function.

The submatrix $[g^{x\varphi}]$ can be obtained by multiplying the coefficients in relations 2.32(b) by the term $2(a_yP + M_x)$.

The coefficients of the *symmetric* submatrix $[g^{y\varphi}]$ may be easily deduced from relations 2.32(b) by *increasing the first index by four* multiplying them by the term $-2(a_xP - M_y)$.

2.10.2.2 Submatrix $[g^{\varphi\varphi}]$

Using Equation 2.34(a), one needs to generate the following coefficients

$$g_{9,9} = -g_{9,11} = g_{11,11} = \frac{\kappa}{D^2 l^3} [(1 - \text{ch}\ \kappa)(3\text{sh}\ \kappa - \kappa) + \kappa\ \text{sh}^2\kappa]$$

$$g_{9,10} = -g_{10,11} = \frac{1}{D^2 l^3} [(4 + \frac{\kappa^2}{2} + \frac{\kappa^2}{2}\ \text{sh}\ \kappa)(1 - \text{ch}\ \kappa) + 2\text{sh}^2\ \kappa]$$

$$g_{10,10} = g_{12,12} = \frac{1}{\kappa l^3 D^2} [(\text{ch}\ \kappa - 1)(\text{sh}\ \kappa + \kappa) + \kappa\ \text{sh}\ \kappa(\kappa - 2\text{sh}\ \kappa) - \frac{\kappa^2}{2} (\kappa - \text{sh}\ \kappa\ \text{ch}\ \kappa)]$$

$$g_{10,12} = \frac{1}{\kappa l^3 D^2} [(\text{sh}\ \kappa - 3\kappa)(1 - \text{ch}\ \kappa) + \frac{\kappa^2}{2} (\kappa\ \text{ch}\ \kappa - 3\text{sh}\ \kappa)] \qquad \ldots\ 2.34(b)$$

and $g_{11,12} = -g_{9,12} = -g_{9,10}$.

The submatrix $[g^{\varphi\varphi}]$ is obtained by multiplying the coefficients given by relations 2.34(b) by the factor $(P r^2 + 2\beta_x M_y - 2\beta_y M_x)$.

2.10.3 Mass Matrix

The mass matrix defined by the *third* integral in Equation 2.24 is of the following form

$$[m_{ij}] = \begin{bmatrix} m^{xx} & 0 & m^{x\varphi} \\ 0 & m^{yy} & m^{y\varphi} \\ m^{\varphi x} & m^{\varphi y} & m^{\varphi\varphi} \end{bmatrix} \qquad \ldots\ 2.35$$

Referring to Przemieniecki (1968), submatrices $[m^{xx}]$ and $[m^{yy}]$, corresponding to transverse vibrations in the two principal planes, can be represented in the non-dimensional form as

$$[m^{xx}] = \frac{\rho A l}{420} \begin{bmatrix} 156 & -22 & 54 & 13 \\ -22 & 4 & -13 & -3 \\ 54 & -13 & 156 & 22 \\ 13 & -3 & 22 & 4 \end{bmatrix} \quad \ldots \; 2.36(a)$$

and

$$[m^{yy}] = \frac{\rho A l}{420} \begin{bmatrix} 156 & 22 & 54 & -13 \\ 22 & 4 & 13 & -3 \\ 54 & 13 & 156 & -22 \\ -13 & -3 & -22 & 4 \end{bmatrix} \quad \ldots \; 2.36(b)$$

As the submatrices $[m^{\varphi x}]$ and $[m^{\varphi y}]$ are transposes of $[m^{x\varphi}]$ and $[m^{y\varphi}]$ respectively, using the third integral of Equation 2.24, one needs to evaluate submatrices $[m^{x\varphi}]$, $[m^{y\varphi}]$ and $[m^{\varphi\varphi}]$ only. The elements of these *symmetric* submatrices are

$$m_{ij}^{x\varphi} = 2\,\rho A\, a_y \int_0^l \gamma_i \gamma_j \, dz \quad \ldots \; 2.37(a)$$

$(i = 1, \ldots, 4,\ j = 9, \ldots, 12)$

$$m_{ij}^{y\varphi} = -\,2\rho A\, a_x \int_0^l \gamma_i \gamma_j \, dz \quad \ldots \; 2.38$$

$(i = 5, \ldots, 8,\ j = 9, \ldots, 12)$

and $$m_{ij}^{\varphi\varphi} = \rho A\, r^2 \int_0^l \gamma_i \gamma_j \, dz \quad \ldots . 2.39(a)$$

$(i, j = 9, \ldots, 12)$

2.10.3.1 Submatrices $[m^{x\varphi}]$ and $[m^{y\varphi}]$

Making use of Equation 2.37(a), it is sufficient to generate the following coefficients:

$$m_{1,9} = m_{3,11} = \frac{1}{lD\kappa^4}\left[(12\kappa - \kappa^3 + \frac{7}{20}\kappa^5)\text{sh}\kappa + (24 + \frac{\kappa^4}{2})(1 - \text{ch}\,\kappa)\right]$$

$$m_{2,9} = -m_{2,11} = -\frac{1}{lD\kappa^4}\left[(6\kappa + \frac{\kappa^5}{20})\text{sh}\kappa + (12 + \kappa^2 + \frac{\kappa^4}{12})(1 - \text{ch}\,\kappa)\right]$$

$$m_{3,9} = m_{1,11} = \frac{1}{lD\kappa^4}\left[(-12\kappa + \kappa^3 + \frac{3}{20}\kappa^5)\text{sh}\kappa + (-24 + \frac{\kappa^4}{2})(1 - \text{ch}\,\kappa)\right]$$

$$m_{4,9} = -m_{4,11} = \frac{1}{lD\kappa^4}\left[(-6\kappa + \frac{\kappa^5}{30})\text{sh}\kappa + (-12 - \kappa^2 + \frac{\kappa^4}{12})(1 - \text{ch}\,\kappa)\right]$$

$$m_{1,10} = -m_{3,12} = \frac{1}{lD\kappa^4}\left[(-6\kappa + \frac{3}{2}\kappa^3)\text{sh}\kappa + (12 - \kappa^2 - \frac{7}{20}\kappa^4)\text{ch}\kappa - 12 + \kappa^2 - \frac{3}{20}\kappa^4\right]$$

... 2.37(b)

$$m_{2,10} = m_{4,12} = \frac{1}{lD\kappa^4}\left[(5\kappa - \frac{\kappa^3}{12})\text{sh}\,\kappa - (8 + \kappa^2 - \frac{\kappa^4}{20})\text{ch}\,\kappa + 8 + \frac{\kappa^4}{30}\right]$$

$$m_{3,10} = -m_{1,12} = \frac{1}{lD\kappa^4}\left[(6\kappa + \frac{\kappa^3}{2})\text{sh}\,\kappa - (12 + \kappa^2 + \frac{3\kappa^4}{20})\text{ch}\,\kappa + 12 + \kappa^2 - \frac{7}{20}\kappa^4\right]$$

$$m_{4,10} = m_{2,12} = \frac{1}{lD\kappa^4} [(\kappa + \frac{\kappa^3}{12})sh\ \kappa - (4 + \frac{\kappa^4}{30})ch\ \kappa + 4 + \kappa^2 - \frac{\kappa^4}{20}]$$

The *symmetric* submatrix $[m^{x\varphi}]$ can be obtained by multiplying the coefficients in relations 2.37(b) by the term $2\rho\ A\ a_y$.

The coefficients of the symmetric submatrix $[m^{y\varphi}]$ may be easily deduced from relations 2.37(b) by increasing the first index by four and multiplying them by $-2\rho\ A\ a_x$.

2.10.3.2 Submatrix $[m^{\varphi\varphi}]$

Using Equation 2.39(a), one needs to generate the following coefficients

$$m_{9,9} = m_{11,11} = \frac{1}{l^2 D^2 \kappa} [(-5sh\ \kappa + 2\kappa - \kappa\ ch\ \kappa + \kappa^2 sh\ \kappa)(1 - ch\,\kappa) + (\frac{\kappa^3}{3} - 2\ \kappa)\ sh^2\kappa]$$

$$m_{9,10} = \frac{1}{l^2 D^2 \kappa^2} [(\frac{7}{2}\ \kappa sh\ \kappa - \frac{\kappa^2}{2})(1 - ch\kappa) + 2\kappa^2 sh^2\ \kappa - \frac{\kappa^3}{6}\ sh\ \kappa(1 + 2ch\ \kappa)]$$

$$m_{9,11} = 0.5 - m_{9,9}$$

$$m_{9,12} = \frac{1}{l^2 D^2 \kappa^2} [-(8 + \frac{5\kappa}{2}\ sh\kappa + \frac{\kappa^2}{2})(1 - ch\ \kappa) - 4sh^2\kappa + \frac{\kappa^3}{6}\ sh\ \kappa\ (2 + ch\ \kappa) - \kappa^2 sh^2\kappa]$$

$$m_{10,10} = m_{12,12} = \frac{1}{l^2 D^2 \kappa^3} \Big[(3\kappa + 3\mathrm{sh}\,\kappa)(1 - \mathrm{ch}\,\kappa) + \frac{\kappa^3}{6}(7 + 2\mathrm{ch}\,\kappa)$$

$$- \kappa^2 \mathrm{sh}\,\kappa(2 + \frac{5}{2}\mathrm{ch}\,\kappa) + \underline{(6\kappa + \frac{\kappa^3}{3})\,\mathrm{sh}^2\,\kappa}\Big]$$

... 2.39(b)

It may be noted that after checking the expression for the coefficient $m_{10,10}$, as given by Krajcinovic (1969), the term in last parantheses in the original expression was seen to be incorrect. The form appearing above is the *amended* one, the correction being indicated by the underscoring. Continuing

$$m_{10,11} = \frac{1}{l^2 D \kappa^2} \Big[(2 - \frac{\kappa^2}{2})(1 - \mathrm{ch}\,\kappa) + \kappa(2\mathrm{sh}\kappa - \kappa\,\mathrm{ch}\,\kappa)\Big] - m_{9,10}$$

$$m_{10,12} = \frac{1}{l^2 D^2 \kappa^3} \Big[(5\kappa + \frac{2}{3}\kappa^3 - 3\mathrm{sh}\,\kappa)(1 - \mathrm{ch}\,\kappa) - \frac{\kappa^3}{2}\mathrm{ch}\,\kappa$$

$$+ \kappa^2 \mathrm{sh}\,\kappa(\frac{7}{2} + \mathrm{ch}\,\kappa) - \kappa^3 - (2\kappa + \frac{\kappa^3}{6})\,\mathrm{sh}^2\kappa\Big]$$

and

$$m_{11,12} = \frac{1}{l^2 D \kappa^2} \Big[-(2 + \frac{\kappa^2}{2})(1 - \mathrm{ch}\,\kappa) + \kappa(\kappa - 2\mathrm{sh}\,\kappa)\Big] - m_{9,12}$$

... 2.39(b)

The submatrix $[m^{\varphi\varphi}]$ is obtained by multiplying the coefficients given in relations 2.39(b) by the factor $\rho A r^2$.

It may be noted that, in the final account, all three matrices (stiffness, stability and mass) are symmetric, reflecting the fact that the problem is self-adjoint. All

derived matrices are computed in *local co-ordinate system*. In order to form the *global* matrix, necessary *topological (transformation)* matrices should be used in the same way as for solid beam assemblages.

2.11 SOLUTION OF THE PROBLEM

Equation 2.26 is general enough to briefly discuss a host of distinct problems – such as static analysis, elastic instability, free steady state vibrations etc. It is emphasised that the *displacement method* is used as the *basis* for the analysis of any of these problems and Equation 2.26 enables one to evaluate the *nodal displacements* $\{q_i\}$, being considered as *unknowns* in the problem.

2.11.1 Static Analysis

The governing equilibrium equation is

$$\{F_i\} = [k_{ij}] \{q_i\} \qquad \ldots 2.40$$

where $\{F_i\}$ is the vector of *external equivalent nodal loads* so that

$$\{q_i\} = [k_{ij}]^{-1} \{F_i\}. \qquad \ldots 2.41$$

Having generated vector $\{F_i\}$ from the *known* external loading and matrix $[k_{ij}]$ assembled for the structure, from the known properties of the component thin-walled members, one can determine the unknown displacements, using Equation 2.41, for a given set of boundary conditions.

This enables one to work out the *internal forces* at any section i.e. the generalized nodal forces which are expressible in terms of the nodal displacements, now known. The *stresses* and *deformations* at any section can then be worked out using the *internal force-stress* and *internal force-deformation* relationships which take into account the *properties* of the section. In other words, this approach essentially means the *stiffness method* of analysis.

2.11.2 Structural Instability

Assuming that all external *(conservative)* forces are characterized by a single parameter say λ, one can write for the homogeneous case, from Equation 2.25 as

$$([k_{ij}] - \lambda \ [g_{ij}]) \{q_i\} = 0 \qquad \ldots \ 2.42$$

Equation 2.42 represents a well-known eigenvalue problem wherein the nontrivial solution for $\{q_i\}$ is obtained if λ is an eigenvalue of the matrix $[g_{ij}]^{-1} \ [k_{ij}]$. The buckling load calculated from the smallest eigenvalue λ_{min} should be an *upperbound* to the exact solution as a consequence of the applied *variational principle*.

2.11.3 Vibration Analysis

The general vibration problem (neglecting damping) of small oscillations about the equilibrium position is governed by Equation 2.25. For the steady state solution to the problem, the solution can be sought in the form

$$q_j(t) = q_j \exp(i\,\omega t) \qquad \ldots\ 2.43$$

where 'ω' is the natural frequency of the system and 'i' the imaginery unit. Hence, Equation 2.25 can be written for the homogeneous case as

$$(-\,\omega^2\,[m_{ij}] + [k_{ij}] - [\tilde{g}_{ij}])\,\{q_i\} = 0, \qquad \ldots\ 2.44$$

the curl denoting that the premultiplying scalar is not considered.

The natural frequencies are now the *eigenvalues* of the matrix $[m_{ij}]^{-1}\,([k_{ij}] - [\tilde{g}_{ij}])$. For the same reason as in the previous case, the obtained natural frequencies are upper bounds to the exact solution.

2.12 A LIMITING CASE

Although the analysis of an assemblage of thin-walled members according to the procedure presented, does not involve special computational difficulties, it is possible to simplify it further in certain cases depending on the member geometry. The differential Equation 2.4(d) governing the problem of the torsion differs only by *its second term* on the left hand side from Equations 2.4(b) and (c), governing the flexure of the beam. The magnitude of this term is, furthermore, dependent on the parameter k (or in non-dimensional form κ), being the square root of the ratio of *torsional rigidity* (GK) and *warping rigidity* ($EI_{\Omega\Omega}$). It is, therefore, of

interest to analyze the influence of this term on relations derived heretofore. The analysis of a limiting case $\kappa \to 0$ will not only give simple formulae but will provide the possibility of a *qualitative analysis* of the problem, in the sense that a very simple relation between the torsion and bending can be established.

When the torsional rigidity can be neglected in comparison with the warping rigidity ($GK << EI_{\Omega\Omega}$, i.e. $kl \to 0$), the second term in Equation 2.4(d) indicating the effect of the St. Venant's torsion can be discarded. Thus Equation 2.4(d) is simplified to

$$EI_{\Omega\Omega}\, \varphi'''' = m_t + m_\Omega' \qquad \ldots\ 2.45$$

The solution of the homogeneous equation 2.45 is obviously a cubic polynomial. Moreover the displacement modes γ_i^φ have to be identical with corresponding modes for bending γ_i^x. Using an adequate number of terms of series expansions for hyperbolic trigonometric functions 'sh' and 'ch', the displacement function $\gamma_9(z)$ may be written as

$$\gamma_9(z) = D_1/D \qquad \ldots\ 2.46(a)$$

where

$$D_1 = \frac{\kappa^4}{12} - \frac{\kappa^2(kz)^2}{4} + \frac{\kappa(kz)^3}{6} + \ldots \qquad \ldots\ 2.46(b)$$

$$\text{and} \quad D = 2(1 - \text{ch}\,\kappa) + \kappa\,\text{sh}\,\kappa \qquad \ldots\ 2.15$$

On expanding this becomes

$$D = 2(1 - 1 - \frac{\kappa^2}{2} - \frac{\kappa^4}{24} - \ldots) + \kappa\ (\kappa + \frac{\kappa^3}{6} + \ldots\ )$$

$$= \frac{\kappa^4}{12} + \ldots . \quad \text{neglecting the higher order terms.}$$

Thus in the limiting case considered

$$\lim_{\kappa \to 0} \gamma_9(z) = 1 - \frac{3z^2}{l^2} + \frac{2z^3}{l^3} = \gamma_1(z)$$

In exactly the same way it may be shown that

$$\lim \gamma_{10}(z) = \gamma_2(z); \quad \lim \gamma_{11}(z) = \gamma_3(z) \text{ and } \lim \gamma_{12}(z) = \gamma_4(z)$$

as κ approaches zero.

As a result of this, for $\kappa \approx 0$, the torsional moment can be evaluated as a transverse force and the bimoment as a bending moment, if the external torsional load is considered as a transverse load and the bimoment as distributed couples.

In a similar manner, it may also be proved that when κ approaches zero

$$\lim_{\kappa \to 0} [k^{\varphi\varphi}] = \frac{I_{\Omega\Omega}}{l^2 I_{xx}} [k^{xx}] = [\tilde{k}^{xx}] \quad , \quad \text{say.}$$

Similarly,

$$\lim_{\kappa \to 0} [\tilde{g}^{x\varphi}] = \lim_{\kappa \to 0} [\tilde{g}^{\varphi\varphi}] = [\tilde{g}^{xx}]$$

and,

$$\lim_{\kappa \to 0} [\tilde{m}^{x\varphi}] = \lim_{\kappa \to 0} [\tilde{m}^{\varphi\varphi}] = [\tilde{m}^{xx}]$$

This means that all matrices differ only by a multiplicator having the same form as for solid beams. Hence, this limiting case, besides being a useful and simple approximation (when applicable), justifies entirely the procedure presented as being nothing but an *extension* of established procedures for the solid beams.

2.13 CONCLUDING REMARKS

The presented matrix formulation for the static and dynamic analysis of the structures assembled from thin-walled members is by itself quite *general* and *consistent*. A number of related linearized problems such as lateral buckling, second order theory, parametric resonance etc. may also be treated once the stiffness, stability and mass matrices for the structure are assembled using the relations established earlier.

However, one important point may be emphasized in this regard. In case of linear static problems, the solution presented herein is exact since the displacement modes γ_i are the *exact* solutions of the governing homogeneous Equations 2.4(b) and (c). For the dynamic and stability problems, the story is somewhat *different*. The choice of displacement modes is a matter of assumptions and thus, subject to discussion. In the present case it is assumed that the ***static*** deformation modes are *similar* in shape to *buckling* and *dynamic* modes. Quite often this assumption was

successfully employed in conjunction with variational methods, long before the advent of the finite element method. However, as a consequence, the *buckling load* and the *natural vibrational frequency* obtained using this formulation are expected to be *upperbounds* to the *exact* solutions.

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3. OPTIMAL DESIGN OF THIN-WALLED SECTIONS SUBJECT TO STATIC CONSTRAINTS

3.1 BRIEF LITERATURE SURVEY

3.1.1 Analysis of Thin-Walled Sections

Before the publication of the classical treatise on thin-walled elastic beams by Vlasov (1959), attempts **were made** by Patterson (1952) and Nylander (1956) to study combined flexural torsion and lateral buckling of I-shaped beams of mono-symmetrical cross-section.

El Darwish (1965) provided charts and empirical formulae for the accurate calculation of torsion constant for different structural shapes of open cross-section. Nethercot (1974) proposed a relation, based on experimental results, to determine the torsional rigidity of I-sections, and concluded that the presence of residual stresses increases it significantly. Marshall (1971) presented a basis for determining displacements and stresses arising from torsion of structural rectangular hollow sections, with particular attention to stress concentrations at the re-entrant corners.

Horne (1971) studied the behaviour of thin-walled prismatic members of doubly symmetric open cross-section in which the second moment of area about the major principal axis is very much greater than that about the minor principal axis. The governing equations for the member were derived and the solution obtained.

Conway (1972) proposed a very simple formula for calculating the stress at the re-entrant corner of a twisted structural angle or channel.

Chang (1975) derived a method of computing the warping shear stresses due to restraints in a thin-walled beam of open cross-section, using finite elements.

Holden (1972) presented the numerical solution of three different types of problems of finite deflection of uniform thin beams using the Bernoulli-Euler relation.

3.1.2 Optimization of Thin-Walled Sections

Not much work appears to have been done on the optimal designs of thin-walled structures.

Haaijer (1961) obtained optimum proportions that result in the maximum strength or stiffness for a given cross-sectional area in I-shaped homogeneous beams of high strength steel. Schilling (1974) extended this work on hybrid beams and gave equations and curves showing the variation of bending strength and stiffness, and other related properties with various geometric parameters for both hybrid and homogeneous beams.

Barnett (1961) developed design procedures for minimum weight, statically determinate beams using a deflection criterion. Optimum flange width and web thickness variations were described for I-beams designed with respect to bending and shear deflection.

Dupuis (1971) pointed out that the optimal design of beams subject to deflection constraint alone might not be a well posed problem. As such, he introduced an upper bound on the bending stress as an additional constraint and further showed that these two behavioural constraints were complementary.

Felton (1971) and Nelson (1972) considered basic aspects of the design of indeterminate structures composed of thin-walled beam elements. The minimum weight design of such elastic framed structures was obtained, subject to constraints against two types of local instability as well against yield. The solutions were examined numerically by using a non-linear programming technique.

Bronowicki (1975) pointed out that the optimal design formulation of the type above, although infeasible in practice, provided a lower bound on obtainable weight. After reformulation and utilizing a mathematical programming search technique, he obtained optimal designs of potentially more practical utility in the case of continuous thin-walled beams. The set of behavioural constraints was expanded to include effects of shear stress on web buckling, a factor which was not considered in the previous studies.

However, all these papers have not considered the effect of warping of the section and the stresses associated with it, although this is a chharacteristic feature of a thin-walled beam. In obtaining the optimal solutions for I-shaped sections, the transverse loading is considered to be

in the web-plane. In practice, this being seldom possible, the torsion created is likely to produce the warping stresses of such magnitude worthy to be considered in the design.

Hence, in the present thesis, these foregoing stresses have been *regarded* in various behavioural constraints considered for the optimal design of I-sections. In addition to the usual in-plane loading, the one having lateral eccentricity with respect to the web plane is also included to consider torsional effects.

Consequently, the discrete element analysis employed considers *four* degrees of freedom at each node of the beam element, as against the usual *two*. The additional two degrees are the *longitudinal twist* and the *warping* of the section, the internal forces corresponding to these being the *warping torsion* and the *bimoment*, which together represent the effect of flexural torsion produced in any thin-walled beam member. The optimal solutions obtained incorporating such provisions in analysis and design are therefore expected to be more rational.

3.2 DESIGN VARIABLES

As it is proposed to obtain the minimum weight design by keeping the section *constant* throughout the member span, the problem indirectly reduces to optimization of the cross-sectional dimensions. Referring to Figure 3.1(a), the given section can be looked upon as an assemblage of three

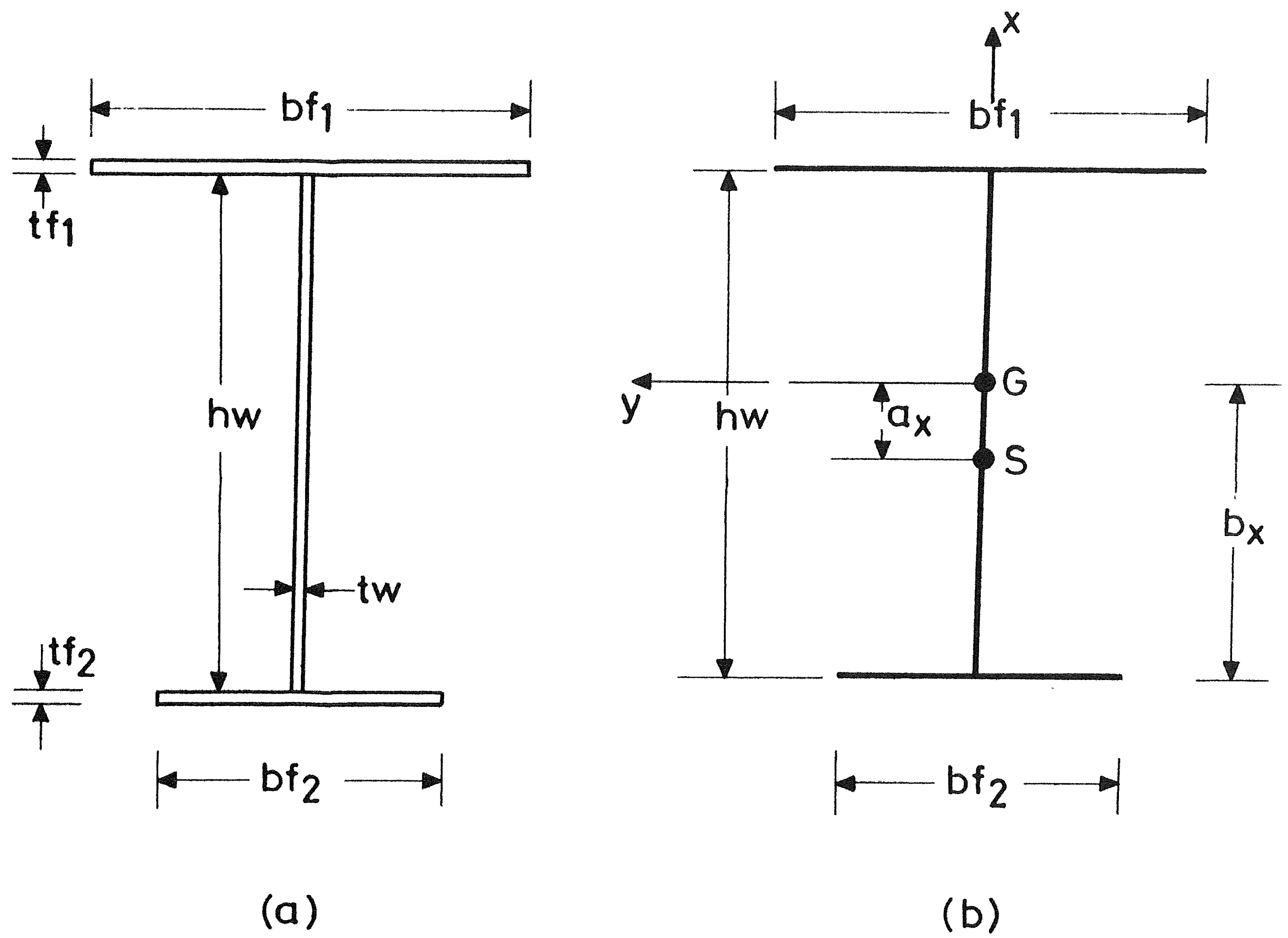


Fig. 3.1 Idealized cross-section.

thin plates: the top flange plate of size (bf_1, tf_1), the bottom flange plate of size (bf_2, tf_2) and these two being connected by the web plate of size (hw, tw). These variables representing cross-sectional dimensions can be viewed as to constitute the design vector

$$\bar{X} = \{x_i\} = \begin{Bmatrix} bf_1 \\ tf_1 \\ bf_2 \\ tf_2 \\ hw \\ tw \end{Bmatrix} \qquad \ldots \; 3.1$$

The plate elements being thin-walled, the section can be *idealized* and considered for analysis in the form shown in Figure 3.1(b). It is aimed at maintaining the sectional symmetry with respect to the minor principal axis, so that the centroid G and the shear centre S will always lie on the x-axis. With these assumptions, various properties of the section, with G as the origin of co-ordinates x and y, can be written as [Figure 3.1(b)].

Sectional area, $A = bf_1 . tf_1 + bf_2 . tf_2 + hw . tw \qquad \ldots \; 3.2(a)$

Position of the centroid G,

$$b_x = \frac{hw . tw . \frac{hw}{2} + bf_1 . tf_1 . hw}{A} \qquad \ldots \; 3.2(b)$$

3.3 DISCRETE ELEMENT ANALYSIS

Consider a thin-walled beam of I-section carrying any transverse loading and subjected to any boundary conditions [Fig. 3.2(a)]. The loading is considered to lie in the web-plane [Fig. 3.2(b)], although in some cases it can as well be eccentric with respect to the web plane [Fig. 3.2(c)]. Adopting a two-element discretization of the structure and noting that the degree of freedom at each node is four, the generalized unknown nodal displacements and the corresponding nodal forces are as shown in Figure 3.3.

The displacement vector $\{q_i\}$ for the *structure* can be written down in the form of subvectors $\{q^A\}$, $\{q^B\}$ and $\{q^C\}$ such that

$$\{q_i\} = \begin{Bmatrix} q^A \\ q^C \\ q^B \end{Bmatrix} \qquad \ldots\ 3.4$$

where $$\{q^A\} = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} = \begin{Bmatrix} \xi_A \\ \xi'_A l \\ \varphi_A l \\ \varphi'_A l^2 \end{Bmatrix} \qquad \ldots\ 3.5(a)$$

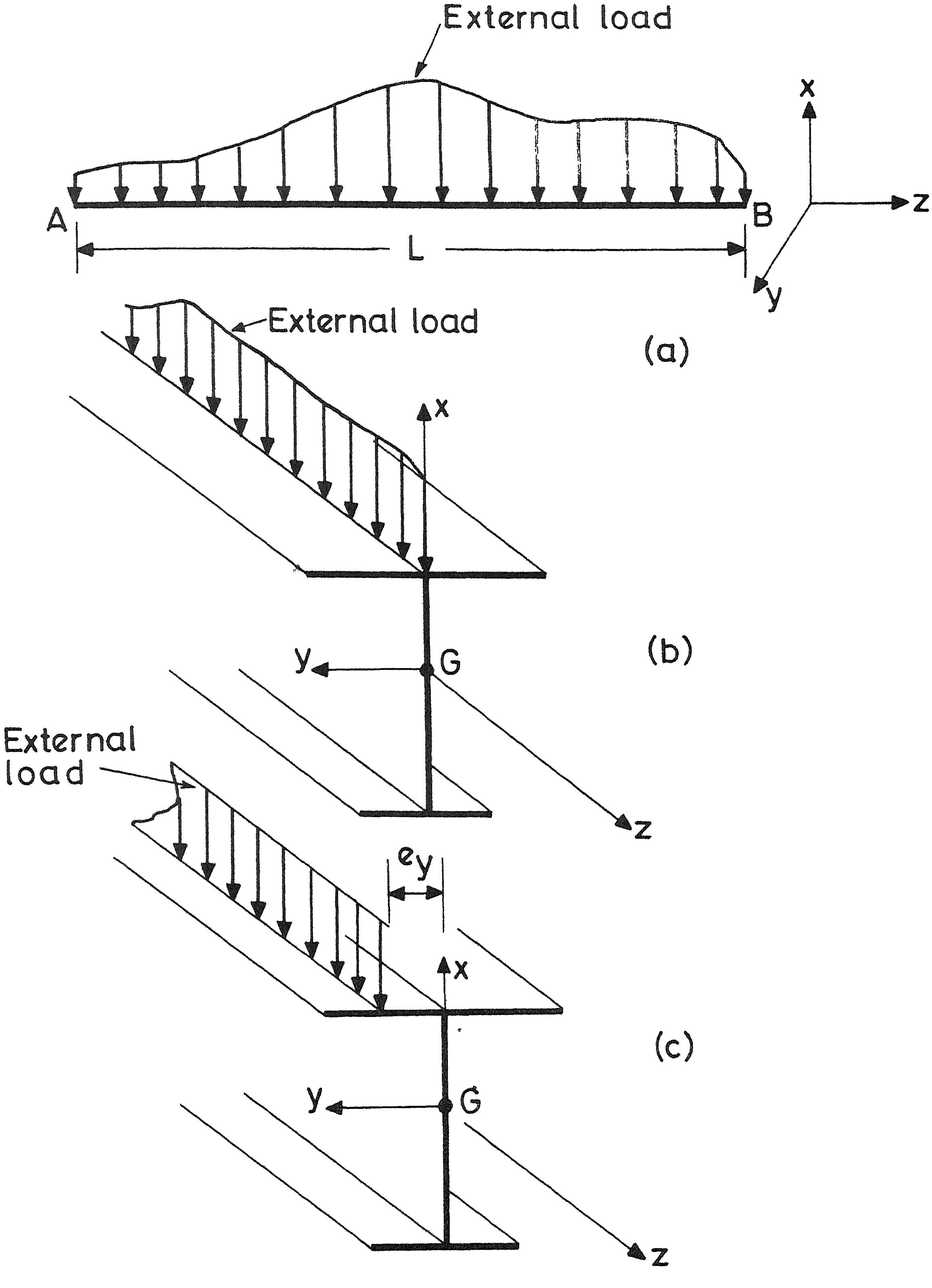
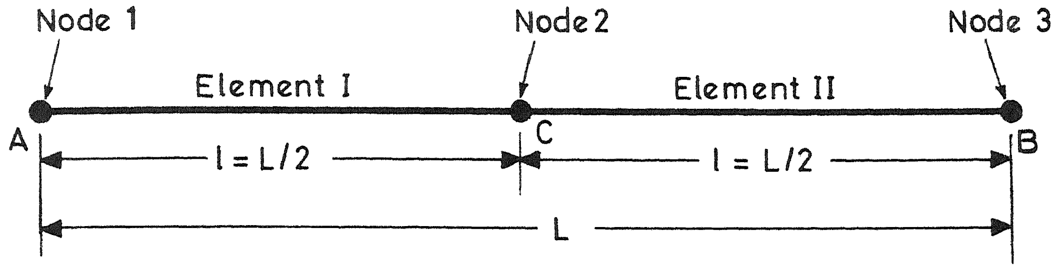
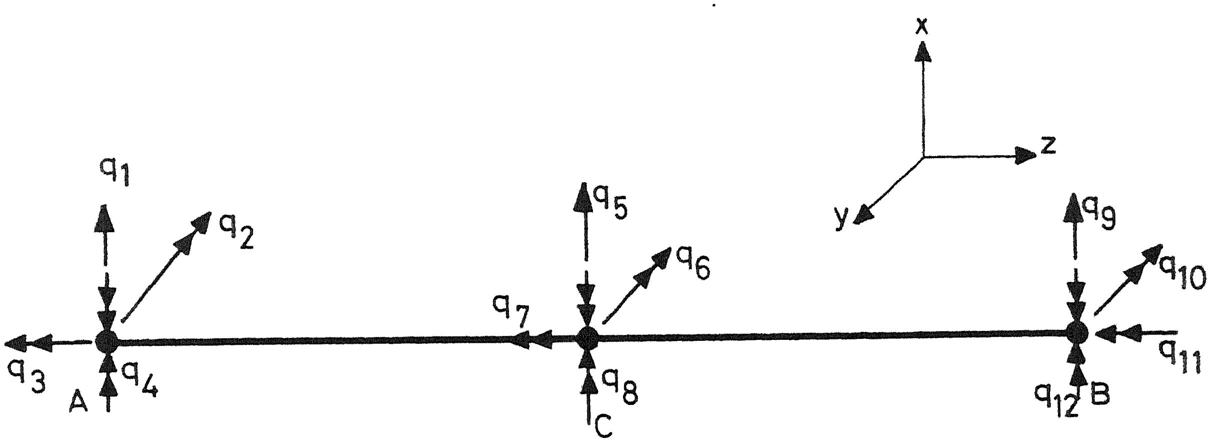


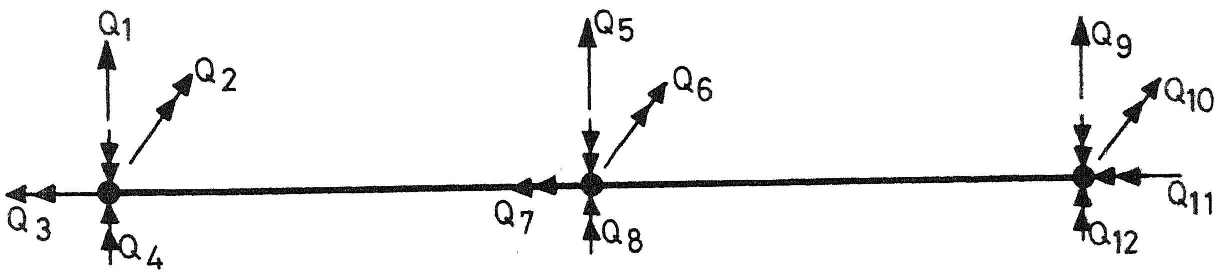
Fig. 3.2 Transverse loading on a thin-walled beam.



(a) Two-element idealization



(b) Generalized nodal displacements



(c) Generalized nodal forces

Fig. 3.3 Static analysis of a thin-walled beam.

$$\{q^C\} = \begin{Bmatrix} q_5 \\ q_6 \\ q_7 \\ q_8 \end{Bmatrix} = \begin{Bmatrix} \xi_C \\ \xi'_C l \\ \varphi_C l \\ \varphi'_C l^2 \end{Bmatrix} \qquad \ldots\ 3.5(b)$$

and,

$$\{q^B\} = \begin{Bmatrix} q_9 \\ q_{10} \\ q_{11} \\ q_{12} \end{Bmatrix} = \begin{Bmatrix} \xi_B \\ \xi'_B l \\ \varphi_B l \\ \varphi'_B l^2 \end{Bmatrix} . \qquad \ldots\ 3.5(c)$$

The corresponding nodal force vector $\{Q_i\}$ for the *structure* is similarly a combination of subvectors $\{Q^A\}$, $\{Q^B\}$ and $\{Q^C\}$ such that

$$\{Q_i\} = \begin{Bmatrix} Q^A \\ Q^C \\ Q^B \end{Bmatrix} \qquad \ldots\ 3.6$$

where

$$\{Q^A\} = \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix} = \begin{Bmatrix} V_{xA} \\ M_{yA}/l \\ T_A/l \\ M_{\Omega A}/l^2 \end{Bmatrix} \qquad \ldots\ 3.7(a)$$

$$\{Q^C\} = \begin{Bmatrix} Q_5 \\ Q_6 \\ Q_7 \\ Q_8 \end{Bmatrix} = \begin{Bmatrix} V_{xC} \\ M_{yC}/l \\ T_C/l \\ M_{\Omega C}/l^2 \end{Bmatrix} \qquad \ldots\ 3.7(b)$$

and

$$\{Q^B\} = \begin{Bmatrix} Q_9 \\ Q_{10} \\ Q_{11} \\ Q_{12} \end{Bmatrix} = \begin{Bmatrix} V_{xB} \\ M_{yB}/l \\ T_B/l \\ M_{\Omega B}/l^2 \end{Bmatrix} \qquad \ldots\ 3.7(c)$$

It may be noted that the displacements $\{q^A\}$ and $\{q^B\}$ are governed by the *prescribed boundary conditions.*

3.3.1 Equivalent Nodal Loads

The governing equilibrium equation for calculating the nodal displacements in a structure is

$$\{F_i\} = [k_{ij}]\,\{q_i\} \qquad \ldots\ 2.40$$

where $\{F_i\}$ is the vector of *external equivalent nodal loads* corresponding to the displacements $\{q_i\}$, given by Equation 3.4.

In general, the actual loads on a structure may not be *necessarily* acting at the nodes. Hence these are divided into two types: loads acting at the *nodes* and loads acting

on the *elements* of the discretized structure. Loads of the latter type are replaced by their statically equivalent loads obtained as the *fixed end reactions* and written as the vector {R}. These reactions when *reversed* and added to the *actual* nodal loads, generate the vector $\{F_i\}$.

For the types of loads considered in the problem, one can readily make use of the relations given in Gere (1969) to obtain the fixed end reactions, provided the loads lie in the web plane. If they are eccentric [Figure 3.2(c)], these produce a varying torsion along the length of the beam. In order to determine the fixed end reactions namely, the twisting moment and the bimoment produced in such a case, attention is drawn to Article 2.12, with particular reference to the simplification achieved in evaluating these, when the ratio k^2 (warping rigidity/torsional rigidity) is very small. In the present problem the value of k^2 is seen to be in the range of 2×10^{-5} to 4×10^{-5}, *small enough* to be neglected. Hence, the fixed end twisting moment and the bimoment can be evaluated as the transverse force and the bending moment respectively, considering the external torsional load as the transverse load and the bimoment as distributed couples.

Thus, obtaining the unknown nodal displacements from the relation

$$\{q_i\} = [k_{ij}]^{-1} \{F_i\} \qquad \ldots\ 2.41$$

the vector of the *internal forces* or *generalized nodal forces* can be determined from

$$\{Q_i\} = \{R\} + [k_{ij}] \{q_i\} \qquad \ldots 3.8$$

Again, if the effect of the St. Venant torsion (T_s) is neglected, Equation 2.1 yields $T_\Omega \approx T$.

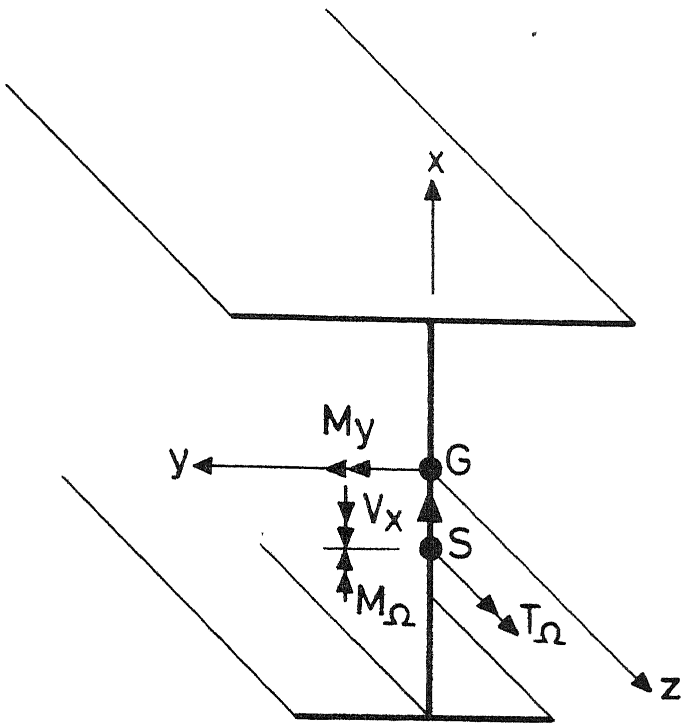
3.3.2 Internal Nodal Stresses

Having obtained the four internal forces, V_x, M_y/l, T_Ω/l and M_Ω/l^2 at every node, one can obtain the internal stresses at a node.

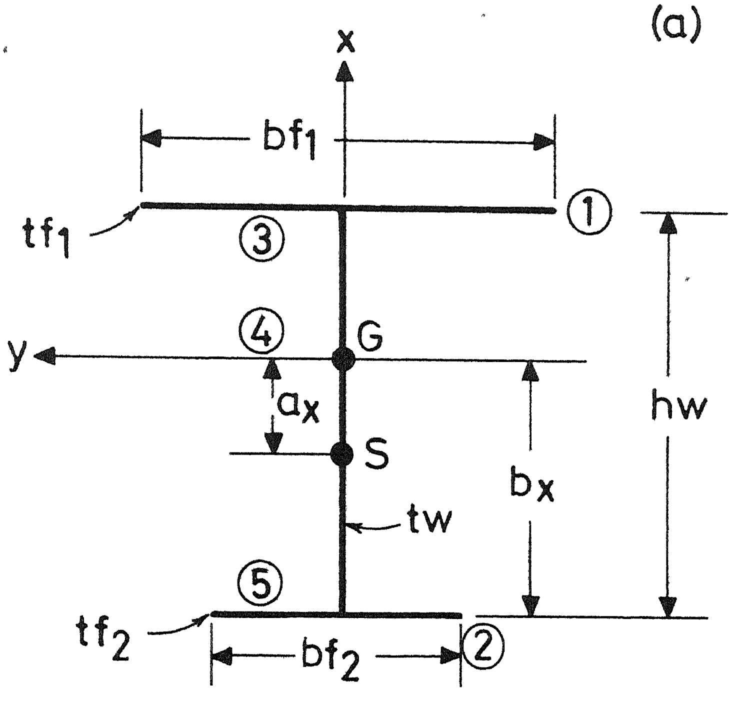
As mentioned in Article 2.3, a loading state at a section in a thin-walled beam can be considered as a superposition of two independent states — the flexure and the flexural torsion. As such, the total stress at any point can be summed up as due to the internal forces V_x, M_y (due to flexure) and T_Ω, M_Ω (due to flexural torsion) [Figure 3.4(a)].

The contribution to the normal stress and the shear stress at any point due to the flexural torsion is governed by the following two theorems of Vlasov:

(a) The stress in a longitudinal fibre of a thin-walled beam due to a bimoment is equal to a product of this bimoment multiplied by the principal sectorial co-ordinate of this fibre divided by the principal sectorial moment of inertia of its cross-section.



(a)



(b)

Fig. 3.4 Internal forces in a thin-walled section.

(b) A shear stress in a fibre of a thin-walled beam caused by a flexural twist is equal to the product of the flexural twist multiplied by the sectorial statical moment of this point divided by the wall thickness (at this point) and the principal sectorial moment of inertia.

Hence, in the present case of an I-section, combining the effects of the flexure and the flexural torsion, the expressions for the maximum normal stress and the maximum shear stress at a node can be written as follows [Figure 3.4(b)].

3.3.2.1 Normal stress

The maximum normal stress is expected to occur either at point (1) or (2) in the *flange*, in which case its values are

$$\sigma^{(1)} = \frac{M_y.(hw - b_x)}{I_{xx}} + \frac{M_\Omega \cdot \frac{bf_1}{2}(hw - b_x + a_x)}{I_{\Omega\Omega}}$$

$$\sigma^{(2)} = \frac{M_y.b_x}{I_{xx}} + \frac{M_\Omega \cdot \frac{bf_2}{2}(b_x - a_x)}{I_{\Omega\Omega}} \qquad \ldots 3.9$$

3.3.2.2 Shear stress

The maximum shear stress is expected to occur either at point (3), (4) or (5) in the *web* and in the respective case, its value is given by

$$\tau^{(1)} = \frac{V_x.[bf_1.tf_1(hw - b_x)]}{tw \;.\; I_{xx}} + \frac{T_\Omega.[\frac{bf_1^2}{8} \;.\; tf_1(hw - b_x + a_x)]}{tw \;.\; I_{\Omega\Omega}}$$

$$\tau^{(2)} = \frac{V_x.[bf_1.tf_1(hw - b_x) + \frac{tw}{2}(hw - b_x)^2]}{tw \;.\; I_{xx}}$$

$$\tau^{(3)} = \frac{V_x.(bf_2.tf_2.b_x)}{tw \;.\; I_{xx}} + \frac{T_\Omega.[\frac{bf_2^2}{8} \;.\; tf_2(b_x - a_x)]}{tw \;.\; I_{\Omega\Omega}}$$

... 3.10

It may be emphasised that the second part in Equations 3.9 and 3.10 is the *stress increase in comparison to the conventional solid beam theory*, due to the effect of warping of the section which is a characteristic feature of a thin-walled member.

Thus, at any node i, the maximum normal and shearing stresses are

$$\sigma_i^{max} = \max\ [\sigma^{(1)},\ \sigma^{(2)}] \qquad \text{... 3.11(a)}$$

$$\tau_i^{max} = \max\ [\tau^{(1)},\ \tau^{(2)},\ \tau^{(3)}] \qquad \text{... 3.11(b)}$$

and in the *whole domain* of the beam, the *critical* values of the normal and the shearing stresses, to be considered for the optimal design having a *uniform* section are

$$\sigma_{cr} = \max\ [\sigma_i^{max}] \qquad (i = 1,...,3) \qquad \text{... 3.12(a)}$$

$$\tau_{cr} = \max\ [\tau_i^{max}] \qquad (i = 1,...,3) \qquad \text{... 3.12(b)}$$

for the two-element solution.

As already discussed in this article, once the nodal displacements $\{q_i\}$ are obtained using Equation 2.41, the critical deformations that can be considered in such an optimal design are

$$\begin{aligned}
\xi_{cr} &= \max\ [|q_1,\ q_5,\ q_9\ |] \\
(\xi' 1)_{cr} &= \max\ [|q_2,\ q_6,\ q_{10}|] \\
(\varphi 1)_{cr} &= \max\ [|q_3,\ q_7,\ q_{11}|] \qquad \ldots\ 3.13 \\
\text{and} \quad (\varphi' 1^2)_{cr} &= \max\ [|q_4,\ q_8,\ q_{12}|]
\end{aligned}$$

3.4 CONSTRAINTS IN THE OPTIMIZATION PROBLEM

Out of the sixteen constraints prescribed in the problem, twelve are the *side* constraints, the remaining four being the *behaviour* constraints. Recalling that the design vector is

$$\bar{X} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{Bmatrix} = \begin{Bmatrix} bf_1 \\ tf_1 \\ bf_2 \\ tf_2 \\ hw \\ tw \end{Bmatrix} \qquad \ldots\ 3.1$$

the constraint relations can be obtained as follows.

3.4.1 Side Constraints

3.4.1.1 Lower and upper bounds on the design variables

Using subscripts 'min' and 'max' to mean the minimum and maximum values respectively of the design variables bf_1, tf_1, ... etc., the constraint relations in the *normalized* form are

$$g_1(\bar{X}) = \frac{bf_{1min}}{x_1} - 1 \qquad \text{... 3.14(a)}$$

$$g_2(\bar{X}) = \frac{tf_{1min}}{x_2} - 1 \qquad \text{... 3.14(b)}$$

$$g_3(\bar{X}) = \frac{bf_{2min}}{x_3} - 1 \qquad \text{... 3.14(c)}$$

$$g_4(\bar{X}) = \frac{tf_{2min}}{x_4} - 1 \qquad \text{... 3.14(d)}$$

$$g_5(\bar{X}) = \frac{hw_{min}}{x_5} - 1 \qquad \text{... 3.14(e)}$$

$$g_6(\bar{X}) = \frac{tw_{min}}{x_6} - 1 \qquad \text{... 3.14(f)}$$

$$g_7(\bar{X}) = \frac{x_1}{bf_{1max}} - 1 \qquad \text{... 3.14(g)}$$

$$g_8(\bar{X}) = \frac{x_2}{tf_{1max}} - 1 \qquad \text{... 3.14(h)}$$

$$g_9(\bar{X}) = \frac{x_3}{bf_{2max}} - 1 \qquad \text{... 3.14(i)}$$

$$g_{10}(\bar{X}) = \frac{x_4}{tf_{2max}} - 1 \qquad \ldots\ 3.14(j)$$

$$g_{11}(\bar{X}) = \frac{x_5}{hw_{max}} - 1 \qquad \ldots\ 3.14(k)$$

$$g_{12}(\bar{X}) = \frac{x_6}{tw_{max}} - 1 \qquad \ldots\ 3.14(l)$$

3.4.2 Behaviour Constraints

3.4.2.1 Maximum normal and shear stresses in the beam

Denoting the permissible normal and shear stresses as σ_{per} and τ_{per} respectively,

$$g_{13}(\bar{X}) = \frac{\sigma_{cr}}{\sigma_{per}} - 1 \qquad \ldots\ 3.14(m)$$

$$g_{14}(\bar{X}) = \frac{\tau_{cr}}{\tau_{per}} - 1 \qquad \ldots\ 3.14(n)$$

3.4.2.2 Maximum deflection in the beam

If ξ_{per} is the permissible deflection in the beam

$$g_{15}(\bar{X}) = \frac{\xi_{cr}}{\xi_{per}} - 1 \qquad 3.14(p)$$

3.4.2.3 Maximum slenderness ratio of the compression flange

If SR_1 is the slenderness ratio of the compression flange with respect to the appropriate radius of gyration, as per the Clause 21.3 of IS : 800 (1962), this value should not exceed 300. So

$$g_{16}(\overline{X}) = \frac{SR_1}{300} - 1 \qquad \ldots\ 3.14(q)$$

3.5 DATA FOR THE ANALYSIS

3.5.1 Number of Elements

As has been established by Krajcinovic (1969), a two-element solution gives quite satisfactory results for the static analysis problems. Hence in the present case of optimization involving static constraints, the analysis using a two-element idealization [Figure 3.3(a)] is adopted.

3.5.2 Boundary Conditions in the Problem

The support conditions chosen are: fixed, simple and one end fixed with the other end free (cantilever). Referring to the generalized nodal displacements shown in Figure 3.3(b), it is understood that a *fixed* boundary will *restrain* all the four displacements at the support node, a *free* boundary will *allow* these to *occur* while a *simple* boundary restrains only the lateral displacement and the longitudinal twist, allowing the transverse rotation and the warping to occur. Depending upon whether a particular displacement at a support is restrained or not, it is assigned a code number unity or zero. The generalized nodal displacements in a two-element discretized model have been numbered [Figure 3.3(b)] and using the coding system mentioned above, any type of boundary condition can be generated.

3.5.3 Elastic Constants

The beam material being structural steel, in accordance with the Clause F-1.1 (Appendix F), IS : 800 (1962), the values of the elastic modulus 'E' and the modulus of rigidity 'G' have been taken as 2.047×10^6 kg/cm^2 and 0.4 E respectively.

3.5.4 Span of the Beam

The range of span L chosen is from 3.0 to 6.0 m for the fixed and the simple supports, and from 1.0 to 2.0 m for the cantilever.

3.5.5 Types and Range of Loading

Both the *nodal* and the *non-nodal* (applied on the elements) loads of the following types have been considered in the respective ranges.

(a) Nodal loads *equivalent* to the uniformly distributed load of intensity range from 1000 to 3000 kg/m.

(b) Uniformly distributed load of intensity range from 500 to 3000 kg/m.

(c) Linearly varying load of intensity range varying from 0-2500 kg/m at one end to 500-3000 kg/m at the other.

(d) Uniformly distributed load of intensity range varying from 1000 to 3000 kg/m, having a *uniform* eccentricity 'e_y' with respect to the web plane. The range of this eccentricity is from 0.1 to 0.3 times the top flange width.

3.5.6 Effective Length of the Compression Flange

In the present problem, it is assumed that no lateral restraint is provided for the compression flange but the beam is *partially* restrained against torsion. Referring to the Clauses 21.2.1 and 21.2.3 of IS : 800 (1962), the coefficient for calculating the effective length of the compression flange is taken as 0.85. The slenderness ratio 'SR_1' of the compression flange thus calculated with respect to its appropriate axis is used in Equation 3.14(q).

3.5.7 Permissible Bending and Shearing Stresses

Using the Clause 10.2.2.1 and Table VII of IS : 800 (1962), the critical stress 'C_s' in the compression element is calculated. The allowable stress in bending 'P_{bc}' as given by Table III of IS : 800 (1962) is 1650 kg/cm^2. The value of the permissible bending stress σ_{per}, to be used in Equation 3.14(m) is the lesser of C_s and P_{bc} above.

Referring to the Clause 10.3.1 and Table VIII of IS : 800 (1962), the value of the permissible shearing stress τ_{per} in Equation 3.14(n) is taken as 1100 kg/cm^2.

3.5.8 Permissible Deflection in the Beam

Referring to the Clause 21.4.1 of IS : 800 (1962), the limiting value of the deflection ξ_{per} to be used in Equation 3.14(p) is $\frac{1}{325}$ of the span, based on the strength or efficiency of the structure or safety against damage to finishing.

3.5.9 Lower Bounds on the Design Variables

Referring to the range of available rolled steel I-sections given in Table I of SP-6(1) (1964), the values of bf_{1min}, bf_{2min}, hw_{min} and tw_{min} to be used in Equations 3.14(a), 3.14(c), 3.14(e) and 3.14(f) are prescribed as 5.0, 5.0, 7.5 and 0.3 cm respectively.

According to the Clause 17.4 of IS : 800 (1962), the controlling thickness for rolled steel being the mean flange thickness, the Clause 17.2 of IS : 800 (1962) fixes the values of tf_{1min} and tf_{2min} to be used in Equations 3.14(b) and 3.14(d), as 0.6 cm.

3.5.10 Upper Bounds on the Design Variables

Referring to the range of available rolled steel I-sections given in Table I of SP-6(1) (1964), the values of bf_{1max}, tf_{1max}, bf_{2max}, tf_{2max}, hw_{max} and tw_{max} are taken as 25.0, 2.5, 25.0, 2.5, 60.0 and 1.25 cm respectively. In addition, following conditions also govern the values of bf_{1max}, bf_{2max} and hw_{max}.

3.5.10.1 Maximum outstand of flange

As per the Clause 21.5.1.3 and Table XVIII of IS : 800 (1962), the maximum outstand of the compression and the tension flange is sixteen and twenty times the respective flange thickness. This indirectly put an upper bound on the values of bf_{1max} and bf_{2max}. The values of bf_{1max} and

bf_{2max} to be used in Equations 3.14(g) and 3.14(i) are the lesser of those given by Articles 3.5.10 and 3.5.10.1 respectively.

3.5.10.2 Maximum depth of the web plate

Referring to the Clause 21.6.1, the minimum thickness of an unstiffened web plate should not be less than hw/85 or in other words for a given value of tw, hw $\not>$ 85 tw. This indirectly puts an upper bound on hw. The value of hw_{max} to be used in Equation 3.14(k) is the lesser of that given by Articles 3.5.10 and 3.5.10.2 respectively.

3.6 OBJECTIVE FUNCTION

Weight being the objective function, for a thin-walled beam member of *uniform* cross-section, the objective function reduces to the expression for the cross-sectional area of the member. Hence

$$f(\bar{X}) = x_1x_2 + x_3x_4 + x_5x_6 \qquad \ldots 3.15$$

3.7 OPTIMIZATION PROBLEM

Thus, the present optimization problem is of *constrained* type, of the form:

to find

$$\bar{X} = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_6 \end{Bmatrix} = \begin{Bmatrix} bf_1 \\ tf_1 \\ \vdots \\ tw \end{Bmatrix} \qquad \ldots\ 3.1$$

so as to minimize the function

$$f(\bar{X}) = x_1 x_2 + \ldots + x_5 x_6 \qquad \ldots\ 3.15$$

and subject to the inequality constraints

$$g_j(\bar{X}) \leq 0 , \qquad j = 1, 2, \ldots, 16 \qquad \ldots\ 3.14$$

The solution is attempted by the interior penalty function method, that is widely used for the constrained minimization problems.

3.8 INTERIOR PENALTY FUNCTION METHOD

In this method, a new function 'ϕ', called the *penalty function,* is constructed by augmenting a *penalty term* to the objective function. The penalty term, involving a parameter r called the *penalty parameter,* is chosen such that its value will be small at points away from the constraint boundaries and will tend to infinity as the constraint boundaries are approached. Hence the value of the ϕ-function also *blows up* as the constraint boundaries are approached. Thus, once the constrained minimization of $\phi(\bar{X}, r)$ is started from any *feasible point* $\bar{X}_1$, the subsequent points

generated will always lie *within* the feasible domain, since the constraint boundaries act as *barriers* during the minimization process.

The ϕ-function used is that defined originally by Carroll (1961)

$$\phi(\bar{X}, r) = f(\bar{X}) - r \sum_{j=1}^{m} \frac{1}{g_j(\bar{X})} \qquad \ldots 3.16$$

It can be seen that the *unconstrained minimum* of the penalty function approaches the *constrained minimum* of the objective function, as $r \to 0$. The *iterative* process to obtain this minimum is as follows.

(i) Start with an initial feasible point $\bar{X}_1$, satisfying all the constraints with *strict inequality* sign, that is, $g_j(\bar{X}_1) < 0$ for $j = 1, 2, \ldots, m$. Set the counter $k = 1$ and choose the initial value of the penalty parameter $r_k > 0$.

(ii) Minimize $\phi(\bar{X}, r_k)$ using any of the unconstrained minimization methods and obtain the solution $\bar{X}_{k+1}$.

(iii) Test whether $\bar{X}_{k+1}$ is the optimum solution of the original problem. If $\bar{X}_{k+1}$ is found to be optimum, terminate the process. Otherwise go to the next step.

(iv) Find the value of the next penalty parameter

$$r_{k+1} = c\, r_k$$

where $c < 1$.

(v) Set the counter $k = k + 1$, take the new starting point $\bar{X}_1 = \bar{X}_k$ and go to step (ii).

Steps (i) through (v) are collectively known as the *sequential unconstrained minimization technique* (SUMT).

In the present case, the value of the initial penalty parameter r_1 is taken as

$$r_1 = \frac{f(\bar{X}_1)}{\left[- \sum_{j=1}^{m} \frac{1}{g_j(\bar{X}_1)}\right]}$$

and $c = 0.1$

these values being found satisfactory.

3.9 UNCONSTRAINED MINIMIZATION

For obtaining the unconstrained minimum of the penalty function, the variable metric method by Davidon-Fletcher-Powell (1959, 1963) is used. This method is the best general purpose and currently available unconstrained optimization technique making use of the function derivatives. It is very powerful, stable and continues to progress towards the minimum even while minimizing highly distorted and eccentric functions. It is basically a descent method that makes use of the gradient vector in finding, at *a point*, *a better search direction* given by the vector

$$\bar{S}_i = \{s_i\} = \begin{Bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{Bmatrix} \qquad \ldots\ 3.17$$

The iterative procedure of this method can be stated by the following algorithm:

(i) Start with an initial point $\bar{X}_1$ and an (n x n) positive definite symmetric matrix $[H_1]$ called the Hessian matrix. Usually $[H_1]$ is taken as the identity matrix $[I]$. Set iteration number as $i = 1$.

(ii) Compute the gradient of the function $\bar{\nabla} f_i$, at the point $\bar{X}_i$ and obtain

$$\bar{S}_i = -[H_i]\ \bar{\nabla} f_i$$

(iii) Find the optimal step length λ_i^* in the direction $\bar{S}_i$ and set

$$\bar{X}_{i+1} = \bar{X}_i + \lambda_i^* \bar{S}_i$$

(iv) Test the new point $\bar{X}_{i+1}$ for optimality. If $\bar{X}_{i+1}$ is optimal, terminate the iterative process. Otherwise go to the next step.

(v) Update the [H] matrix as

$$[H_{i+1}] = [H_i] + [M_i] + [N_i]$$

where $[M_i] = \dfrac{\lambda_i^* \bar{S}_i \bar{S}_i^T}{\bar{S}_i^T \bar{Q}_i}$

$$[N_i] = -\frac{([H_i]\bar{Q}_i)([H_i]\bar{Q}_i)^T}{\bar{Q}_i^T [H_i] \bar{Q}_i}$$

and $\bar{Q}_i = \bar{\nabla} f(\bar{X}_{i+1}) - \bar{\nabla} f(\bar{X}_i)$

$$= \bar{\nabla} f_{i+1} - \bar{\nabla} f_i$$

(vi) Set the new iteration number $i = i + 1$ and go to step (ii).

As the method makes use of the gradient of the function, it is preferable to use the more accurate *cubic interpolation method* for obtaining λ_i^* in step (iii) above.

3.10 COMPUTER PROGRAMME

The computer programme prepared for obtaining the results consists of two programmes:

(i) STTANA - which deals with the static analysis part,

and (ii) OPT2 - which deals with the optimization part, based on the results supplied by STTANA.

The functions performed by various subroutines of the programme STTANA are as follows.

(i) CSPROP -- Calculates the cross-sectional properties of a thin-walled I-section.

(ii) ELSTIF -- Generates the stiffness matrix for a thin-walled beam element.

(iii) ASSBLY -- Assembles the stiffness matrix $[k_{ij}]$ for the structure from the individual element matrices. It further takes out the submatrix corresponding to the *free* nodal displacements in the structure.

(iv) MATINV -- Performs the inversion of a matrix.

(v) STAANA -- For a thin-walled beam of I-section of known sectional dimensions, boundary conditions and loading, it calculates the *maximum* values of the normal stress, the shear stress and displacement in the structure, after performing the discrete element analysis of the structure.

The optimal solution is achieved by the programme OPT2 using following subroutines:

(i) MAIN -- Reads the data for the problem, performs various steps in the optimization process and finally returns the optimal sectional dimensions and the corresponding values of the objective function and the constraints.

(ii) SUMT2 -- Obtains the constrained minimum of a given objective function using the sequential unconstrained minimization technique (SUMT).

(iii) PELTY2 -- Generates the objective function and the penalty function for a design point $\bar{X}$.

(iv) TABLE1 -- Gives the critical stress in the compression element as per Table VII of IS : 800 (1962).

(v) DFP2 -- Obtains the unconstrained minimum of a function by the Davidon-Fletcher-Powell method.

(vi) UNIDR2 -- Minimizes a function in one direction using the method of cubic interpolation.

(vii) GRADF2 -- Gives the gradient vector and its modulus at a given point $\bar{\bar{X}}$ in the function, by using the finite difference technique.

3.10.1 Characteristics of the Programme

The programmes STTANA and OPT2 are quite general and possess the following features:

(a) The programme STTANA can yield the two-, four- and eight-element analysis of thin-walled beams having an I-section, with different spans and boundary conditions, and subjected to a wide variety of loads as detailed in Article 3.5.

(b) By numbering the nodes and generalized displacements at each node, and assigning the respective code number to each displacement, any set of boundary conditions can be generated. Thus, in addition to applying the boundary conditions at the *end* nodes, the *intermediate* nodes can also be subjected to such conditions so that the continuous beams can also be handled by the programme STTANA, using this versatile technique.

(c) In each subroutine of the programme OPT2, care is taken to see that during the optimization process the design point does not enter the infeasible domain. Although theoretically impossible, such type of situation is not uncommon due to the numerical errors associated with a digital computer.

(d) At every important stage in the programme OPT2, precaution is taken to cater for a *bound* design point. In the vicinity of such a point, the constraints naturally 'blow up' and to get rid of the consequent numerical instability, necessary logical provision is made in the programme.

Thus, the programme has been made numerically stable to counter all possible difficult situations created during its execution.

The results for all the cases above have been discussed and the conclusions drawn in Chapter 6.

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4. OPTIMAL DESIGN OF THIN-WALLED SECTIONS SUBJECT TO STABILITY OR DYNAMIC CONSTRAINTS

4.1 BRIEF LITERATURE SURVEY

4.1.1 Analysis of Thin-Walled Sections

Gallagher (1963) suggested a matrix displacement formulation of the beam-column problem and presented a general procedure including instability effects in element force-displacement relationships applicable to both beams and plates. Hartz (1965) presented rapid formulation of stability problems for column and frame type structures and of the beam-column type problem for determining the effects of axial loads on laterally loaded structures.

Gallagher (1967) extended the concepts of the matrix displacement approach to discrete element structural analysis to predict general instability. The influence of element membrane forces on element effective flexure stiffnesses was accounted for and critical applied load intensities were predicted for plate, arch and spherical cap structures. Dawe (1969) applied the method to the buckling analysis of rectangular plates under arbitrary member loading.

Chajes (1965) presented a simple method of accounting for torsional-flexural buckling in case of thin-walled open sections. Based on the use of an interaction type of equation, he suggested a relatively uncomplicated

and straightforward procedure for determining the torsional-flexural buckling load in case of singly-symmetric sections, a grouping that includes most of the commonly used shapes. Kennedy (1972), with the help of experimental results on steel angle and tee struts, advocated the use of a rational buckling analysis in stipulating the maximum width-thickness ratios.

Barsoum (1970) formulated the stiffness matrices for the torsional and lateral stability analysis of structures composed of flexural members by the matrix displacement method.

Kawai (1971) performed finite element analysis of thin-walled sections based on modern engineering theory of beams. Rajasekharan (1971) did the same for thin-walled members of open section. Nethercot (1971) demonstrated the accuracy of the finite element method in dealing with problems of buckling of columns and beams.

Wittrick (1968) and Williams (1969) developed the stiffness matrices and the subsequent computational procedure for the buckling or vibration analysis of thin, flat-walled structures. It was shown that buckling and vibration phenomena for any structure of this type are closely inter-related.

Carnegie (1969) presented a method of matrix displacement analysis, with strong convergence characteristics for the determination of eigenvalues and eigenvectors in vibration problems of continuous systems.

Popelar (1969) investigated the stability of the free transverse vibrations of a simply supported, thin-walled elastic beam with double axes of symmetry, for arbitrary initial velocity distributions.

Mei (1970) derived the stiffness and consistent mass matrices for a thin-walled beam element of open cross-section with non-collinear shear centre and centroid.

4.1.2 Optimization of Thin-Walled Sections

Zarghamee (1970) developed a mathematical programming technique for the minimum weight design of such structures wherein the general instability may be the controlling phenomenon in the design. The method was claimed to be applicable to all structures for which a finite element model can be built.

Khot (1976) presented an optimization method based on optimality criterion for minimum weight of structures with stability constraints. The method was developed in the displacement method of finite element analysis and was programmed for trusses and frames.

Taylor (1968) introduced a method for the optimum design of a vibrating bar with specified minimum cross-section.

Kamat (1973) presented a finite element displacement formulation to maximize the first mode natural frequency of

a vibrating beam of specified volume with elastically restrained ends and resting on a continuous elastic foundation.

Sheu (1968) developed a method for the elastic minimum weight design of a one-dimensional structure that is to have a prescribed fundamental frequency. Similar methods were also suggested by Turner (1967) and McCart (1969).

Fried (1969) proposed gradient methods for finite element eigen problems.

However, all these papers above have not considered the effect of warping of the section, although this is a characteristic feature of a thin-walled beam. In the present analysis, this aspect is considered by giving due regard to the degree of freedom at a section.

4.2 STABILITY ANALYSIS

Consider a prismatic member of length L, having a thin-walled, I-shaped cross-section and subjected to a pair of compressive longitudinal loads 'P' acting along the centroidal axis, as shown in Figure 4.1(a).

As is understood, from the well-known theory of elastic stability given by say, Timoshenko (1961), the member becomes *structurally unstable* at a particular value of load P, called the *critical* or *buckling* load 'P_{cr}'. Adopting a two-element discretization of the member [Figure 4.1(a)], the generalized nodal displacements $\{q_i\}$ and the corresponding internal nodal forces $\{Q_i\}$ can be seen to be as shown in

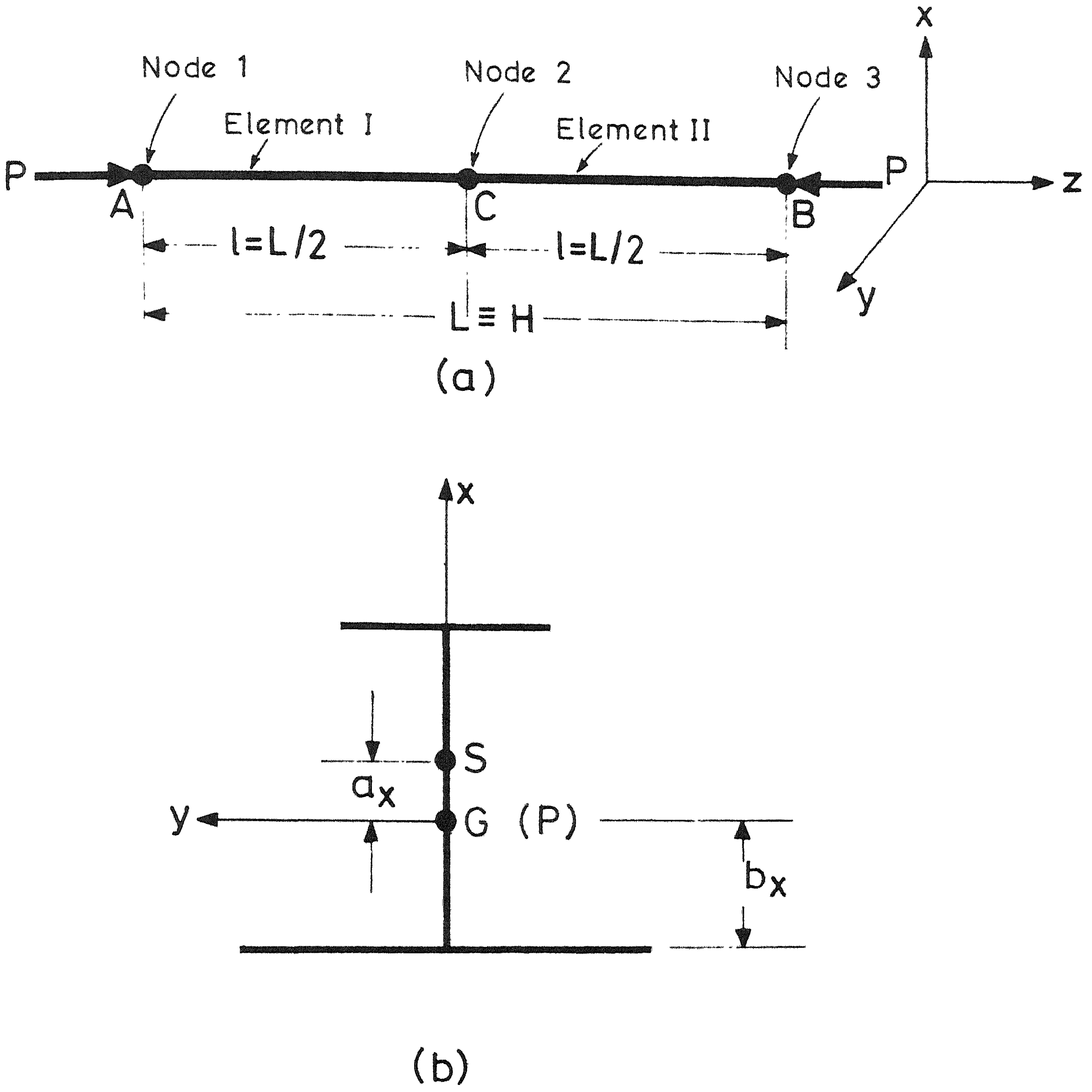

Fig. 4.1 Stability analysis of a thin-walled member.

Figures 3.3(b) and 3.3(c) respectively. It may be noted that $\{q_i\}$ can be expressed in the form of subvectors $\{q_A\}$, $\{q_B\}$ and $\{q_C\}$, as indicated by Equations 3.4 and 3.5, and $\{Q_i\}$ by subvectors $\{Q_A\}$, $\{Q_B\}$ and $\{Q_C\}$, as shown by Equations 3.6 and 3.7. Obviously $\{q_A\}$ and $\{q_B\}$ are governed by the prescribed boundary conditions.

Recalling from Article 2.11.2, for a given set of boundary conditions, the governing equation to determine the *unknown* displacements is the eigenvalue problem

$$([k_{ij}] - \lambda\ [g_{ij}])\ \{q_i\} = 0 \qquad \ldots\ 2.42$$

$[k_{ij}]$ and $[g_{ij}]$ being the stiffness and the stability matrix respectively of the member. The non-trivial solution for $\{q_i\}$ can be obtained by determining λ as an eigenvalue of the matrix $[g_{ij}]^{-1}\ [k_{ij}]$. The value of λ is *directly* related to the critical load P_{cr}, while the vector $\{q_i\}$ obtained corresponding to a value of λ represents the particular *buckling mode* of the member.

4.2.1 Buckling Modes

Three possible modes exist in which centrally loaded columns can buckle:-

(a) they can bend in the plane of one of the principal axes *(flexural mode)*,

(b) they can twist about the shear centre axis *(torsional mode)*,

or, (c) they can bend and twist simultaneously
(torsional-flexural mode).

For any given member, depending on its length and the geometry of its cross-section, *one* of these three modes will be critical.

Referring to Chajes (1965), for an axially loaded column of thin-walled cross-section and having any boundary conditions, the critical load P_{cr} is given by the roots of the general equation in 'P', for the torsional-flexural buckling

$$r^2(P - P_x)(P - P_y)(P - P_\varphi) - P^2 a_y^2 (P - P_y) - P^2 a_x^2 (P - P_x) = 0 \qquad \text{... 4.1}$$

where the parameters P_x, P_y and P_φ have the form of the two pure buckling loads *in* planes of the principal axes and a purely torsional buckling load *about* the shear centre axis. They are given by

$$P_x = \frac{\pi^2 EI_{xx}}{L^2} \qquad \text{... 4.2(a)}$$

$$P_y = \frac{\pi^2 EI_{yy}}{L^2} \qquad \text{... 4.2(b)}$$

and

$$P_\varphi = \frac{1}{r^2}\left(GK + \frac{EI_{\Omega\Omega}\,\pi^2}{L^2}\right) \qquad \text{... 4.2(c)}$$

where L is the effective length of the member, I_{xx} and I_{yy} are given by Equation 2.3(a), K by 2.3(b), $I_{\Omega\Omega}$ by 2.3(c), and r^2 by 2.22(a).

In the present case, the section is symmetric with respect to one axis (x-axis) so that setting $a_y = 0$ reduces Equation 4.1 to

$$(P - P_x)\left[r^2(P - P_y)(P - P_\varphi) - P^2 a_x^2\right] = 0 \qquad \ldots 4.3$$

There are again three solutions, one of which is $P = P_x$ which represents *purely flexural buckling* in x-z plane [Figure 4.1(a)]. The other two are the roots of the quadratic term inside the square brackets equated to zero and give *two torsional-flexural* buckling loads. The *lowest* torsional-flexural load will always be *below* P_y or P_φ. It may, however, be above or below P_x depending upon the sectional dimensions.

As mentioned by Chajes (1965), for sections having $I_{xx} > I_{yy}$, the lowest torsional-flexural buckling load is always less than P_x. Referring to Table I of SP : 6(1) (1964), it may be seen that the entire range of available rolled steel I-sections belongs to the above category and hence the governing buckling mode will be that in torsional flexure. This also emphasises the need to consider the sectional warping in the analysis, that arises as a result of the flexural torsion.

In order to determine the buckling load $P = P_{cr}$ from Equation 2.42, the elements of matrices $[k_{ij}]$ and $[g_{ij}]$, for an element, can be written in the *non-dimensional form* taking out terms $\frac{EI_{yy}}{l^3}$ and $\frac{P}{l}$, outside these matrices respectively. It may be noted that elements of $[k_{ij}]$ correspond to

flexure in the y-z planes [Figure 4.1(a)] and the term $\frac{EI_{yy}}{l^3}$ ensures to give the lowest buckling load, I_{yy} being less than I_{xx}. Thus, for an element in Figure 4.4(a), Equation 2.42 can be written as

$$\left(\frac{EI_{yy}}{l^3}[k_{ij}] - \frac{P}{l}[g_{ij}]\right)\{q_i\} = 0 \qquad \ldots 4.4$$

This can be expressed as the *general* eigenvalue problem, as suggested by Equation 2.42

$$[k_{ij}]\{q_i\} = \lambda [g_{ij}]\{q_i\} \qquad \ldots 4.5$$

where $$\lambda = \frac{Pl^2}{EI_{yy}} \qquad \ldots 4.6$$

It may be noted that the matrices $[k_{ij}]$ and $[g_{ij}]$ above, are *square, positive definite* and *symmetric*.

The *general* eigenvalue problem is solved after converting it into the *standard* form, given by

$$[B]\{q_1\} = \lambda \{q_1\} \qquad \ldots 4.7$$

where $$[g_{ij}] = [L][L]^T \qquad \ldots 4.8(a)$$

obtained after the standard Cholesky decomposition, and then

$$[B] = [L^{-1}][k_{ij}][L^{-1}]^T \qquad \ldots 4.8(b)$$

and $$\{q_1\} = [L]^T \{q_i\} \qquad \ldots 4.8(c)$$

4.2.1.1 Number of critical loads

If 'ndf' is the number of degrees of freedom in the system, as governed by the boundary conditions, the elements of matrices $[k_{ij}]$ and $[g_{ij}]$ in Equation 4.5 correspond to the *free* unknown nodal displacements. Hence these matrices are of the order (ndf x ndf), while the vector $\{q\}$ has size (ndf x 1). The eigenvalue problem has 'ndf' solutions, each corresponding to a critical load on the column. It may be noted that all solutions are *positive* by virtue of the positive definiteness of matrices $[k_{ij}]$ and $[g_{ij}]$.

If λ_{min} is the smallest eigenvalue, the critical load to be considered for the design is given by

$$P_{cr} = \lambda_{min} \frac{EI_{yy}}{l^2} \qquad \ldots 4.9$$

and the corresponding value of buckling stress being critical, is

$$\sigma_{bcr} = \frac{P_{cr}}{A} \qquad \ldots 4.10$$

4.3 CONSTRAINTS IN THE OPTIMIZATION PROBLEM

Out of the fourteen constraints prescribed in the problem, *twelve* are the side constraints, the remaining two being the behaviour constraints.

Recalling that the design vector is

$$\bar{X} = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_6 \end{Bmatrix} = \begin{Bmatrix} bf_1 \\ tf_1 \\ \vdots \\ tw \end{Bmatrix} \qquad \ldots\ 3.1$$

the constraint relations can be obtained as follows.

4.3.1 Side Constraints

4.3.1.1 Lower and upper bounds on the design variables

The twelve relations representing the lower and upper bounds on the design variables are the same as those given by Equations 3.14(a) through 3.14(l).

4.3.2 Behaviour Constraints

4.3.2.1 Maximum slenderness ratio in the member

If 'SR_2' is the slenderness ratio of the column corresponding to the appropriate radius of gyration of the section, in accordance with the Clause 19.2.1 and Table XVI of IS : 800 (1962), this ratio should not exceed 180. So

$$g_{13}(\bar{X}) = \frac{SR_2}{180} - 1 \qquad \ldots\ 4.11(a)$$

4.3.2.2 Maximum compressive stress in the column

Inspection of Equation 4.10 shows that, the maximum compressive stress that one may expect at the working loads is

4.4.4 Height of the Column

The range of column heights chosen is from 3.0 to 6.0 m for the fixed supports, 3.0 to 4.5 m for the simple supports and from 3.0 to 4.0 m for the cantilever.

The *effective* height can be calculated by multiplying the actual height H by a coefficient, whose value in the respective cases above is 0.67, 1.0 and 2.0, as given by Table XV of IS : 800 (1962). The appropriate value of slenderness ratio 'SR_2', thus calculated, is used in Equation 4.11(a).

4.4.5 Lower and Upper Bounds on the Design Variables

The values of bf_{1min}, tf_{1min}, ..., tw_{min} in Equations 3.14(a) through 3.14(f), and bf_{1max}, tf_{1max}, ..., tw_{max} in Equations 3.14(g) through 3.14(l) are the same as given in Articles 3.5.9 and 3.5.10 respectively.

4.4.6 Maximum Outstand of Flanges

As per the Clause 19.4.1 of IS : 800 (1962), the maximum outstand of the compression and the tension flange is sixteen times the respective flange thickness. This condition indirectly sets an upper bound on the values of bf_{1max} and bf_{2max}. The values of bf_{1max} and bf_{2max} to be used in Equations 3.14(g) and 3.14(i) are the lesser of those given by Articles 4.4.5 and 4.4.6 respectively.

4.4.7 Maximum Depth of the Web Plate

As per the Clause 19.4.2.1 of IS : 800 (1962), in computing the effective area and radius of gyration, the depth of the plate should be reckoned as not more than fifty times the web plate thickness. In a minimum weight design, the total depth of web plate hw may be subjected to this limit. The value of hw_{max} to be used in Equation 3.14(k) is the lesser of that given by Articles 4.4.5 and 4.4.7 respectively.

4.4.8 Permissible Value of the Compressive Stress at Buckling

The value of 'FS' to be used in Equation 4.11(b) is 4 and the value of σ_{bper} in Equation 4.11(c) is given by the value of 'P_c' obtained by using Table II of IS : 800 (1962).

4.5 OBJECTIVE FUNCTION

The objective function is given by

$$f(\bar{X}) = x_1x_2 + x_3x_4 + x_5x_6 \qquad \ldots 3.15$$

4.6 OPTIMIZATION PROBLEM

The present constrained optimization problem is given as,

find

$$\bar{X} = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_6 \end{Bmatrix} = \begin{Bmatrix} bf_1 \\ tf_1 \\ \vdots \\ tw \end{Bmatrix} \qquad \ldots\ 3.1$$

so as to minimize the function

$$f(\bar{X}) = x_1x_2 + \ldots + x_5x_6 \qquad \ldots\ 3.15$$

subject to the inequality constraints

$$g_1(\bar{X}) = \frac{bf_{1min}}{x_1} - 1 \leq 0 \qquad \ldots\ 3.14(a)$$

$$g_2(\bar{X}) = \frac{tf_{1min}}{x_2} - 1 \leq 0 \qquad \ldots\ 3.14(b)$$

. . . .

$$g_{12}(\bar{X}) = \frac{x_6}{tw_{max}} - 1 \leq 0 \qquad \ldots\ 3.14(l)$$

$$g_{13}(\bar{X}) = \frac{SR_2}{180} - 1 \leq 0 \qquad \ldots\ 4.11(a)$$

and, $$g_{14}(\bar{X}) = \frac{\sigma_{bw}}{\sigma_{bper}} - 1 \leq 0 \qquad \ldots\ 4.11(c)$$

The solution of this problem is sought by using the interior penalty function method, as discussed in Articles 3.8 and 3.9.

4.7 COMPUTER PROGRAMME

The computer programme prepared for obtaining the results comprises the following two programmes;

(i) STBANA -- which deals with the stability analysis part,

and, (ii) OPT3 -- which deals with the optimization part based on the results supplied by STBANA.

The functions performed by various subroutines of the programme STBANA are as follows:

(i) CSPROP -- Calculates the cross-sectional properties of a thin-walled I-section

(ii) ELSTIF -- Generates the stiffness matrix for a thin-walled beam element

(iii) ELSTAB -- Generates the stability matrix for a thin-walled beam element

(iv) ASSBLY -- Assembles the stiffness matrix $[k_{ij}]$ and the stability matrix $[g_{ij}]$ from the individual element matrices. It further isolates submatrices from $[k_{ij}]$ and $[g_{ij}]$, corresponding to the free displacements in the structure

(v) EIGSTD -- Converts the general eigenvalue problem into the standard one

(vi) CHOSKY -- Performs the Cholesky decomposition of a symmetric, positive definite, square matrix.

(vii) MATMLT -- Obtains the product of two matrices

(viii) MATINV -- Performs the matrix inversion

(ix) EIGEN -- Finds the eigenvalues of a real, symmetric matrix by the Givens-Householder method

and,

(x) STBANA -- For a thin-walled beam of I-section of known sectional dimensions and boundary conditions, it calculates the least value of the critical load, and the corresponding critical stress at buckling, by using a discrete element analysis of the structure.

The optimal solution is obtained by the programme OPT3 using following subroutines:

(i) MAIN -- Reads the data for the problem, performs various steps in the optimization process and finally returns the optimal sectional dimensions and the corresponding values of the objective function and the constraints

(ii) SUMT3 -- Obtains the constrained minimum of a given objective function using the sequential unconstrained minimization technique (SUMT)

(iii) PELTY3 -- Generates the objective function and the penalty function for a design point $\bar{X}$

(iv) TABLE2 -- Gives the allowable stress in axial compression as per Table II of IS : 800 (1962)

(v) DFP3 -- Obtains the unconstrained minimum of a function by the Davidon-Fletcher-Powell method

(vi) UNIDR3 -- Minimizes a function in one direction using the method of cubic interpolation

(vii) GRADF3 -- Gives the gradient vector and its modulus at a given point $\bar{X}$ in the function, by using the finite difference technique.

4.8 DYNAMIC ANALYSIS

Consider a prismatic member of length L having a thin-walled, I-shaped cross-section and subjected to a given set of boundary conditions. Assuming that the member is displaced laterally by a small magnitude from its initial equilibrium position, it will vibrate in transverse direction.

Adopting a two-element discretization of the member [Figure 3.3(a)], the generalized nodal displacements $\{q_i\}$ [Figure 3.3(b)] are expressible as sub-vectors $\{q^A\}$, $\{q^B\}$ and $\{q^C\}$ by Equations 3.4 and 3.5, where $\{q^A\}$ and $\{q^B\}$ are governed by the prescribed boundary conditions.

Recalling from Article 2.11.3, for a given set of boundary conditions and with no axial load 'P' on the member, the governing Equation 2.44 is reduced to

$$([k_{ij}] - \omega^2 [m_{ij}]) \{q_i\} = 0 \qquad \ldots 4.12$$

where $[k_{ij}]$ and $[m_{ij}]$ are the stiffness and the mass matrix respectively. Equation 4.12 represents a *general* eigenvalue problem that gives the *natural transverse vibrational frequencies* 'ω' and the corresponding nodal displacements

$\{q_i\}$, assumed as unknowns in the problem. One may note the similarity between Equations 2.42 and 4.12 and conclude that the buckling and vibration phenomena are closely interrelated.

The non-trivial solution for $\{q_i\}$ can be obtained by first determining ω^2 as the eigenvalue of the matrix $[m_{ij}]^{-1}$ $[k_{ij}]$. The vector $\{q_i\}$ obtained *corresponding* to a value ω, represents a particular *vibrating mode* of the structure.

4.8.1 Vibrating Modes

By virtue of the similarity between Equations 2.42 and 4.12, one can conclude with the help of Article 4.2.1 that the section being symmetric about x-axis [Figure 4.1(b)], the three *possible* modes of transverse vibrations are:

(a) a purely *flexural* mode in the x-z plane [Figure 3.3(a)]

and (b) *two* torsional-flexural vibrating modes.

Since the cross-section has $I_{xx} > I_{yy}$, the governing vibrating mode is expected to be a torsional-flexural one.

In order to determine the lowest vibrating frequency, using I_{yy} (being less than I_{xx}) to express the elements of matrices $[k_{ij}]$ and $[m_{ij}]$ in the non-dimensional form, for an element of the discretized model, Equation 4.12 takes the form

$$\left(\frac{EI_{yy}}{l^3}[k_{ij}] - \omega^2\ \rho Al\ [\bar{m}_{ij}]\right)\{q_i\} = 0 \qquad \ldots\ 4.13$$

where 'ρ' is the mass per unit volume of the beam material. This can be expressed as the *general* eigenvalue problem

$$[k_{ij}]\ \{q_i\} = \lambda\ .\ [m_{ij}]\ \{q_i\} \qquad \ldots\ 4.14$$

where

$$\lambda = \omega^2\ .\ \frac{\rho\ Al^4}{EI_{yy}} \qquad \ldots\ 4.15$$

It may be noted that the matrices $[k_{ij}]$ and $[m_{ij}]$ above are *square, positive definite* and *symmetric*.

As discussed in Article 4.2.1, the *general* eigen-value problem given by Equation 4.13 is solved after converting it into the *standard* form.

As mentioned in Article 4.2.1.1, Equation 4.14 has 'ndf' eigenvalues, all being *positive* by virtue of the positive definiteness of matrices $[k_{ij}]$ and $[m_{ij}]$, where 'ndf' is the degree of freedom of the system. Each solution gives a *natural vibrational frequency* and the corresponding *mode of transverse vibration*. Let ω_1 be the smallest value of the frequency (corresponding to the *first* mode) of transverse vibrations.

4.9 CONSTRAINTS IN THE OPTIMIZATION PROBLEM

Out of the fourteen constraints prescribed in the problem, twelve are the side constraints, the remaining two being the behaviour constraints. Rewriting the design vector

$$\bar{X} = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_6 \end{Bmatrix} = \begin{Bmatrix} bf_1 \\ tf_1 \\ \vdots \\ tw \end{Bmatrix} \qquad \ldots\ 3.1$$

the constraint relations can be obtained as follows.

4.9.1 Side Constraints

4.9.1.1 Lower and Upper Bounds on the Design Variables

The twelve relations representing the lower and upper bounds on the design variables are the same as those given by Equations 3.14(a) through 3.14(l).

4.9.2 Behaviour Constraints

4.9.2.1 Lower and Upper Bounds on the First Natural Frequency

If ω_{min} and ω_{max} represent the lower and upper bounds on the first natural frequency,

$$g_{13}(\bar{X}) = \frac{\omega_{min}}{\omega_1} - 1 \qquad \ldots\ 4.16(a)$$

$$g_{14}(\bar{X}) = \frac{\omega_1}{\omega_{max}} - 1 \qquad \ldots\ 4.16(b)$$

4.10 DATA FOR THE ANALYSIS

The number of elements, the boundary conditions in the problem and the values of elastic constants are those given in Articles 4.4.1, 4.4.2 and 4.4.3 respectively.

4.10.1 Span of the Beam

The range of beam spans chosen is from 3.0 to 6.0 m for the fixed and the simple supports and 1.0 to 2.0 m for the cantilever.

4.10.2 Lower and Upper Bounds on the Design Variables

The values of bf_{1min}, tf_{1min}, ..., tw_{min} in Equations 3.14(a) through 3.14(f), and bf_{1max}, tf_{1max}, ..., tw_{max} in Equations 3.14(g) through 3.14(l) are the same as given in Articles 3.5.9 and 3.5.10 respectively.

4.10.3 Maximum Outstand of Flanges

The values of maximum outstand of flanges which set upper bounds on bf_{1max} and bf_{2max}, are those given by Article 4.4.6.

4.10.4 Maximum Depth of the Web Plate

The value of maximum depth of the web plate which sets an upper bound on hw_{max} is that given by Article 4.4.7.

4.10.5 Lower and Upper Bounds on the First Natural Frequency

The range of spans chosen (Article 4.10.1) and the available rolled steel sections as given by Table I of SP : 6(1) (1964), indirectly decide the lower and upper bounds on the first natural frequency. Two different ranges - a narrow and a broad one, are chosen for each type of boundary conditions.

For fixed supports, the values of ω_{min} and ω_{max} are (100, 200) and (50, 300) respectively for the two different ranges. Similar values for the simple supports are (75, 125) and (30, 150) and for the cantilever these are (100, 200) and (50, 450) respectively. These values have been used in Equations 4.16.

4.11 OBJECTIVE FUNCTION

The objective function is given by

$$f(\bar{X}) = x_1x_2 + x_3x_4 + x_5x_6 \qquad \ldots \; 3.15$$

4.12 OPTIMIZATION PROBLEM

The present constrained optimization problem is given by:

to find

$$\bar{X} = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_6 \end{Bmatrix} = \begin{Bmatrix} bf_1 \\ tf_1 \\ \vdots \\ tw \end{Bmatrix} \qquad \ldots \; 3.1$$

so as to minimize the function

$$f(\bar{X}) = x_1x_2 + \ldots + x_5x_6 \qquad \ldots \; 3.15$$

subject to the inequality constraints

$$g_1(\bar{X}) = \frac{bf_{1min}}{x_1} - 1 \leq 0 \qquad \text{... 3.14(a)}$$

$$g_2(\bar{X}) = \frac{tf_{1min}}{x_2} - 1 \leq 0 \qquad \text{... 3.14(b)}$$

. . . .

$$g_{12}(\bar{X}) = \frac{x_6}{tw_{max}} - 1 \leq 0 \qquad \text{... 3.14(l)}$$

$$g_{13}(\bar{X}) = \frac{\omega_{min}}{\omega_1} - 1 \leq 0 \qquad \text{... 4.16(a)}$$

and,

$$g_{14}(\bar{X}) = \frac{\omega_1}{\omega_{max}} - 1 \leq 0 \qquad \text{... 4.16(b)}$$

The solution of this problem is sought by using the interior penalty function method as discussed in Articles 3.8 and 3.9.

4.13 COMPUTER PROGRAMME

The computer programme prepared for obtaining the results comprises the following two programmes:

(i) FRQANA -- which deals with the dynamic analysis part,

and (ii) OPT4 -- which deals with the optimization part based on the results supplied by FRQANA.

The programme FRQANA comprises subroutines CSPROP, ELSTIF, ELMASS, ASSBLY, EIGSTD, CHOSKY, MATMLT, MATINV, EIGEN and FRQANA.

The functions of these subroutines, except those given below, are described in Article 4.7.

ELMASS -- Generates the mass matrix for a thin-walled beam element

ASSBLY -- Assembles the stiffness matrix $[k_{ij}]$ and the mass matrix $[m_{ij}]$ from the individual element matrices. It further isolates submatrices from $[k_{ij}]$ and $[m_{ij}]$ corresponding to the free displacements in the structure.

FRQANA -- For a thin-walled beam of I-section of known sectional dimensions and boundary conditions, it calculates the first natural frequency of transverse vibration, using the discrete element analysis of the structure.

The optimal solution is obtained by the programme OPT4 using subroutines MAIN, SUMT4, PELTY4, DFP4, UNIDR4 and GRADF4. The functions of these subroutines are the same as those of MAIN, SUMT3, PELTY3,..and GRADF3 respectively and described in Article 4.7.

4.14 CHARACTERISTICS OF THE PROGRAMMES

(a) The programmes STBANA and FRQANA are quite general and can yield the two-, four- and eight-element analysis of the thin-walled beams having an I-section, with different spans and boundary conditions.

(b) As discussed in para (b) of Article 3.10.1, the code numbering used to generate any boundary conditions, enables one to handle the continuous beams as well.

(c) Programmes OPT3 and OPT4 incorporate logical provisions mentioned in paras (c) and (d) of Article 3.10.1, so that they are made numerically stable.

The results for both these problems have been discussed and the conclusions drawn in Chapter 6.

- o -

5. OPTIMAL DESIGN OF A CANTILEVER RETAINING WALL

5.1* GENERAL

A wall designed to maintain a difference in the elevations of the ground surfaces on each side of the wall is called a *retaining wall*. The earth, whose ground surface is at the higher elevation is commonly called the *backfill* and the wall is said to retain this backfill. In general, the term backfill refers to the material behind a wall, whether undisturbed ground or fill, that contributes to the pressure against the wall.

The side of the wall adjacent to the backfill is called the *back* and the side which is exposed for most of its height is called the *face*. The bottom surface of the wall is called the *base* or the *footing*. The line of intersection of the back and base is called the *heel*, and the line of intersection of the front of the wall and the base is called the *toe*. Also, a projecting portion on the front of the wall is called the *toe projection* and one on the back, the *heel projection*. The *batter* of the face or back of a retaining wall is its inclination with the vertical.

Earth, which lies above a horizontal plane at the elevation of the top of a wall, is called a *surcharge*. If the ground surface slopes upward from the top of a wall as

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* Articles 5.1 through 5.7, although of elementary nature, are presented here for the sake of completeness.

shown in Figure 5.1(a), the wall is said to have an *inclined* or *sloping surcharge*.

Retaining walls are extensively used in connection with railways, highways, bridges, canals, and many other civil engineering works.

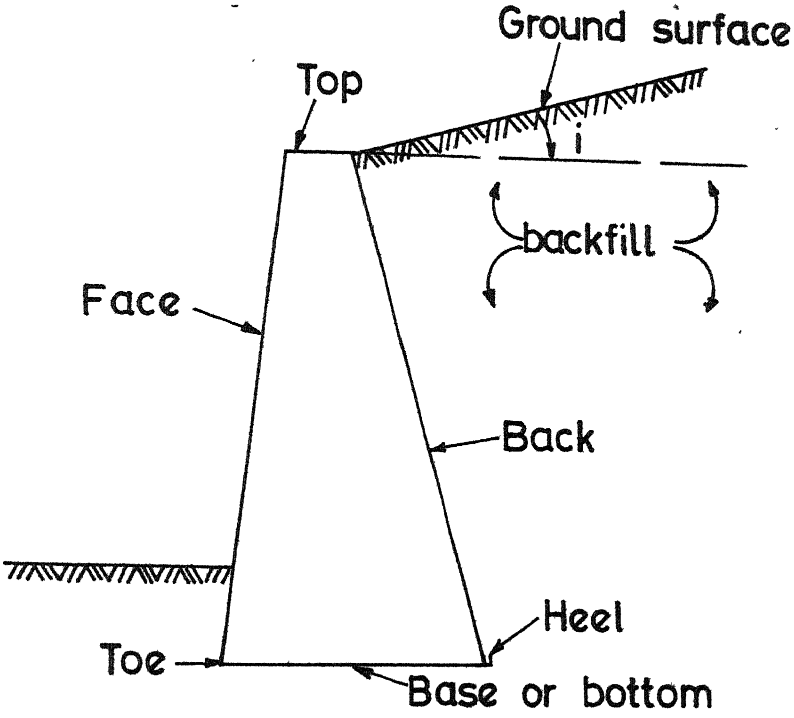
5.2 TYPES OF RETAINING WALLS

The shape of the cross-section, or *profile* of a retaining wall may be determined by the prevailing conditions, but usually retaining walls are of four general types: the solid gravity wall, the semigravity wall, the cantilever or T-wall, and the counterfort wall. The buttressed wall is used occasionally under special conditions. Retaining walls are mostly constructed of plain or reinforced concrete, although plain or brick masonry is sometimes used.

Amongst all these categories, a cantilever type of wall is more widely used than any other type, in spite of the durability advantages of the solid gravity and semigravity walls, because it is usually more economical for ordinary heights, say from 3 to 6 m. A counterfort type of wall may prove to be more economical for greater heights.

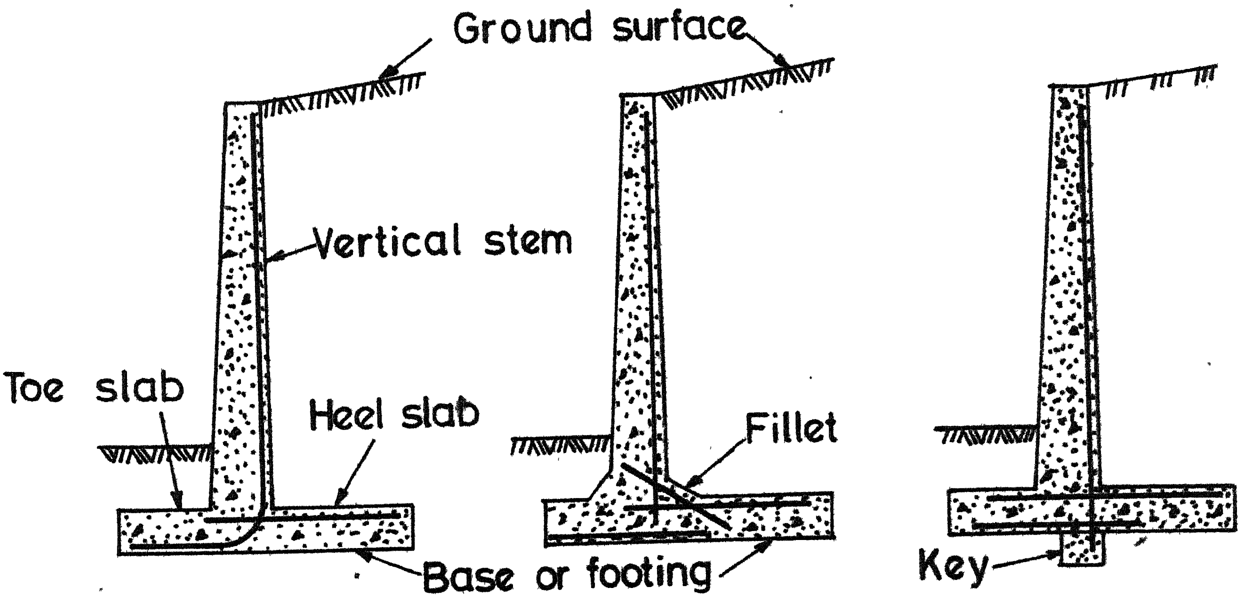
5.3 TYPES OF CANTILEVER WALLS

Common types of cantilever walls are illustrated in Figures 5.1(b), (c) and (d). This type of wall is seen to be made up of three cantilever beams: the vertical stem,



(a)

Simple retaining wall.



(b) (c) (d)

Fig.5.1 Types of cantilever wall.

the toe projection and the heel projection. It is also called a *T-wall* because it resembles an inverted 'T'. The simplest form shown in Figure 5.1(b) is suitable for small heights. For higher walls, it may prove economical to provide *fillets* in the corners as shown in Figure 5.1(c). The sliding resistance may sometimes, but not always, be increased by a downward projection below the base as shown in Figure 5.1(d). This projection, called a *key*, is usually placed near the centre of the base so as to provide end embedment for the vertical reinforcement in the stem. It is more effective in resisting sliding if placed at the heel; it is sometimes placed at the toe as well. A construction joint is provided at the junction of the vertical stem and the top of the footing.

The present optimization problem deals with cantilever walls of ordinary range of heights (3 to 6 m) and so, the type shown in Figure 5.1(b) is chosen for this purpose.

5.4 FORCES ACTING ON RETAINING WALLS

The forces acting on a simple form of retaining wall, where the cross-section remains constant, are shown in Figure 5.2(a). For the sake of simplicity, a trapezoidal wall is shown in the illustration.

The various forces that can be considered for the analysis of a retaining wall are as follows:

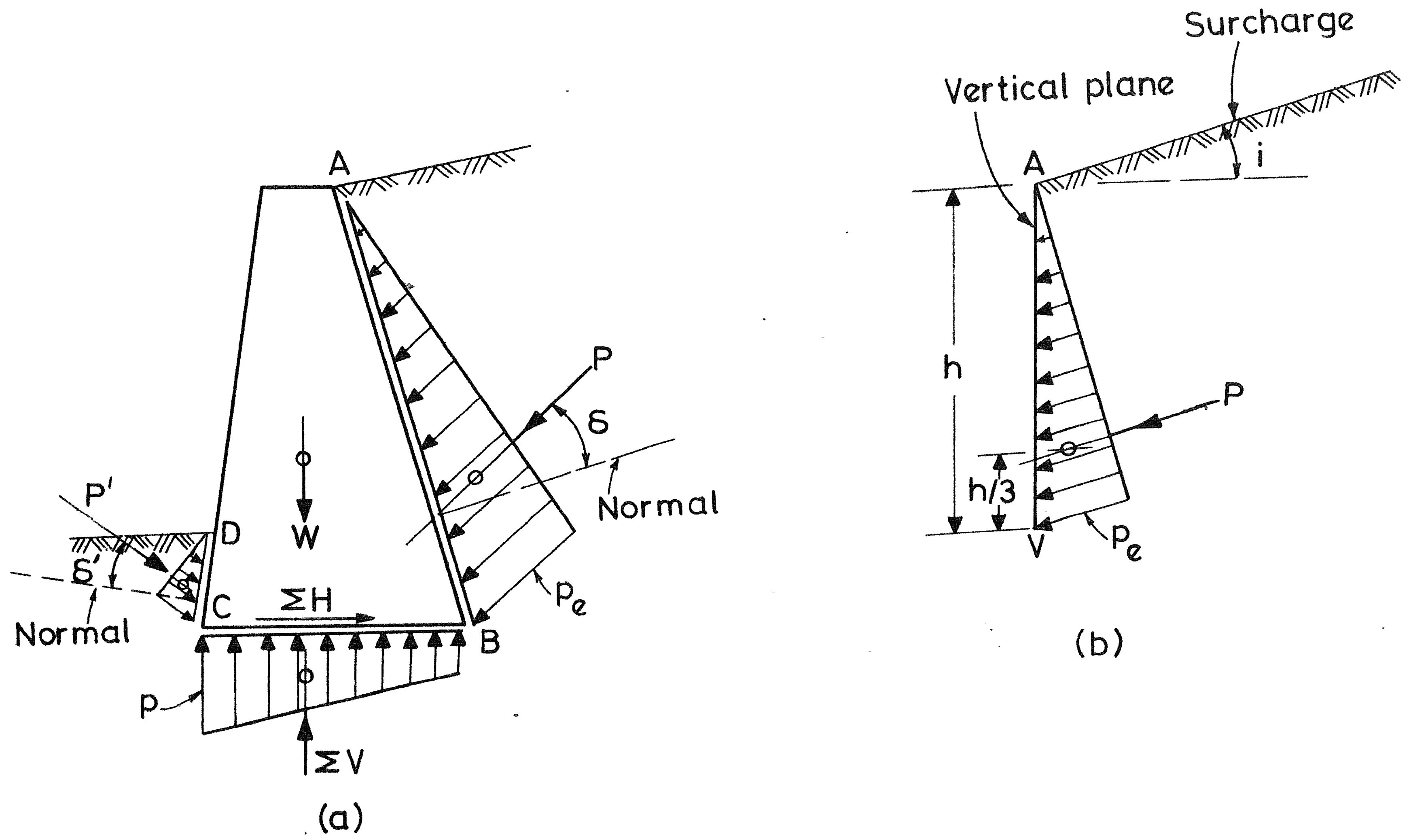


Fig. 5.2 Forces on a retaining wall.

(a) *The weight of the wall.* The resultant of this weight, of course, acts through the centre of gravity of the portion above the horizontal section considered, as indicated by 'W' in Figure 5.2(a). Usually, it is simpler to divide a cross-section into triangles and rectangles, whose areas and centroids can be determined easily and to deal with the corresponding forces separately.

(b) *The earth pressure against the back of the wall.* The pressures usually considered are those due to the lateral thrust of the retained earth, as indicated by the pressure diagram and the resultant force P acting against the back AB with an *obliquity* δ as shown in Figure 5.2(a). This is called *active pressure.*

(c) *The normal components of the foundation pressures.* These are usually considered as being distributed *linearly* across the base CB giving a trapezoidal pressure diagram, as shown in Figure 5.2(a). The resultant of these forces acts through the centroid of the pressure diagram and is indicated by ΣV. The base of a wall is usually horizontal, so that the normal pressures, called the *foundation* or *bearing pressures,* are usually vertical.

(d) *The horizontal components of the foundation pressures.* The resultant of these components is indicated by ΣH in Figure 5.2(a). The *distribution* of these pressures is usually not considered but their *resultant* is considered.

(e) *The earth pressure against the face of the wall.* The bottom of a retaining wall must be placed at a depth wherein the soil with the required bearing capacity can be reached. The earth in front of the wall may exert resistance to forward movement against the buried portion of the wall. This is known as *passive pressure* or *passive resistance* as indicated by the resultant force P' acting at an obliquity δ', as shown in Figure 5.2(a). The force P' adds to the stability of the wall but is omitted from the computations because of uncertainties concerning its magnitude.

The present analysis considers the foregoing forces (a) through (e). However, mention may be made of the following *additional* forces that arise depending on the situation and the purpose for which a retaining wall is constructed:

(f) Bridge reactions, tractive effect and centrifugel force

(g) Surcharge loads

(h) Water and seepage pressures

(i) Uplift

(j) Vibration

(k) Impact

and (l) Earthquake effects.

5.5 PROPERTIES OF SOIL

So far as the earth pressure computations are concerned, the most significant properties of soil are as follows.

(i) *Unit weight of soil (w_s):* With various moisture contents and the degree of compaction.

(ii) *Shearing strength of soil (S):* The shearing strength or resistance of soils is related to the design of retaining walls in two ways: A wall is subjected to lateral earth pressures exerted by a backfill of disturbed soil. The magnitude of these pressures depends upon the shearing resistance. Moreover, the wall is usually supported by a foundation of undisturbed soil. The stability of this support against sliding over the foundation, depends on the shearing resistance. It is therefore necessary to have information concerning the shearing resistance of soils in both the disturbed and undisturbed conditions.

Based on a series of direct shear tests for most soils, the shearing strength may be expressed by the Coulomb's equation

$$S = c + n \tan\phi \qquad \ldots 5.1$$

where S is the unit shearing strength of the soil, n is the unit normal pressure on the surface of failure, c and ϕ are the *cohesion* and the *angle of internal friction* respectively of the soil.

Clean granular materials such as gravel, sand etc., whether saturated or dry, are designated as *cohesionless* and for these, the value of c is taken as zero. Silts and clays are categorized under *cohesive* soils. In the present analysis, while investigating the stability of the wall against sliding, appropriate values of c and ∅ are used in Equation 5.1, taken as the basis.

There are other soil properties like *stress strain relationship, volume change* due to moisture contact, *permeability* etc. but these are not related to the present analysis.

5.6 CLASSES OF EARTH PRESSURES

The *lateral* pressures in a soil mass are greatly affected by any lateral deformation which the mass may have experienced. Such pressures are divided into three classes: earth pressure at rest, or *natural* earth pressure; *active* earth pressure; and *passive* earth pressure.

5.6.1 Natural Earth Pressures

The lateral earth pressures in a natural soil deposit which has remained undisturbed are referred to as *natural earth pressures* or *earth pressures at rest.* Lateral earth pressures in deposits formed artificially by some construction operation and in which no lateral deformation has occurred subsequent to deposition are called

earth pressures at rest. The magnitude of these pressures depends upon the manner in which a soil is deposited as well as upon the physical properties of the soil.

5.6.2 Active Earth Pressures

These exist in a soil deposit which has been *extended* laterally, such as the backfill of a retaining wall, founded on soil which has tilted or moved horizontally a small but sufficient amount due to the pressures exerted by the backfill and the yielding of the foundations.

5.6.3 Passive Earth Pressures

These exist in a soil deposit, which has been *compressed* laterally by the movement of a structure which abuts against it, or in some other manner. The magnitude of the lateral compression of the soil required to develop the *passive* state of stress in a given soil is *somewhat* greater than that of the lateral extension required to produce the *active* state of stress in the soil. Passive pressure is often called *passive resistance.*

5.7 EARTH PRESSURE PHENOMENA

All theories and methods for determining lateral earth pressure due to cohesionless or cohesive soil can be grouped under Rankine's or Coulomb's methods according to their basic assumptions. As given by Huntington (1957),

whether the soil is cohesive or cohesionless, the conditions which govern the application of Rankine's or Coulomb's method may be summarized as follows:

(a) If the back of the wall is a plane surface or a surface which can be assumed to be plane without introducing significant errors, either method may apply.

(b) If the back of the wall cannot be assumed to be a plane surface, Coulomb's condition cannot prevail but Rankine's conditions prevail.

The backs of cantilever and counterfort walls depart so widely from plane surfaces that Rankine's method can be usually applied without significant error, for the cohesionless and the cohesive soils. As mentioned in Article 5.4, the backfill on the retaining wall is under *Rankine's active state of stress* while the soil mass abutting against the face is under *Rankine's passive state of stress* [Figure 5.2(a)]. This *passive resistance* adds to the stability of the wall but is omitted hereafter in the computations because of uncertainties concerning its magnitude.

5.7.1 Active Pressures - Rankine's Conditions

Referring to Figure 5.2(b), the lateral earth pressure p_e in the Rankine's active state of stress and on a vertical plane AV at depth h, is given by Terzaghi (1960) as

$$p_e = w_s h \left[\cos i \cdot \frac{\cos i - \sqrt{\cos^2 i - \cos^2 \phi}}{\cos i + \sqrt{\cos^2 i - \cos^2 \phi}}\right] \quad \ldots\ 5.2$$

$$= w_s h\ K \quad \text{say,}$$

where w_s is the unit weight of soil and i, the inclination of surcharge to the horizontal. K represents the quantity in the parantheses in Equation 5.2.

The resultant active pressure on the vertical plane AV is

$$P = \frac{1}{2} w_s h^2\ K \quad \ldots\ 5.3$$

acting at the *lower third* point of the vertical plane and acting *parallel* to the ground surface.

If the ground surface is horizontal, i = 0 and the factor K in Equation 5.3 reduces to $\frac{1 - \sin\phi}{1 + \sin\phi}$ and

$$P = \frac{1}{2} w_s h^2 \frac{1 - \sin\phi}{1 + \sin\phi} \quad \ldots\ 5.4$$

5.8 PRELIMINARIES IN THE OPTIMIZATION PROBLEM

Consider a typical section of the cantilever retaining wall chosen for the optimization problem, as shown in Figure 5.3(a). The optimization problem is handled on the basis of the following assumptions:

(i) The face of the wall is provided with a batter, keeping the back vertical.

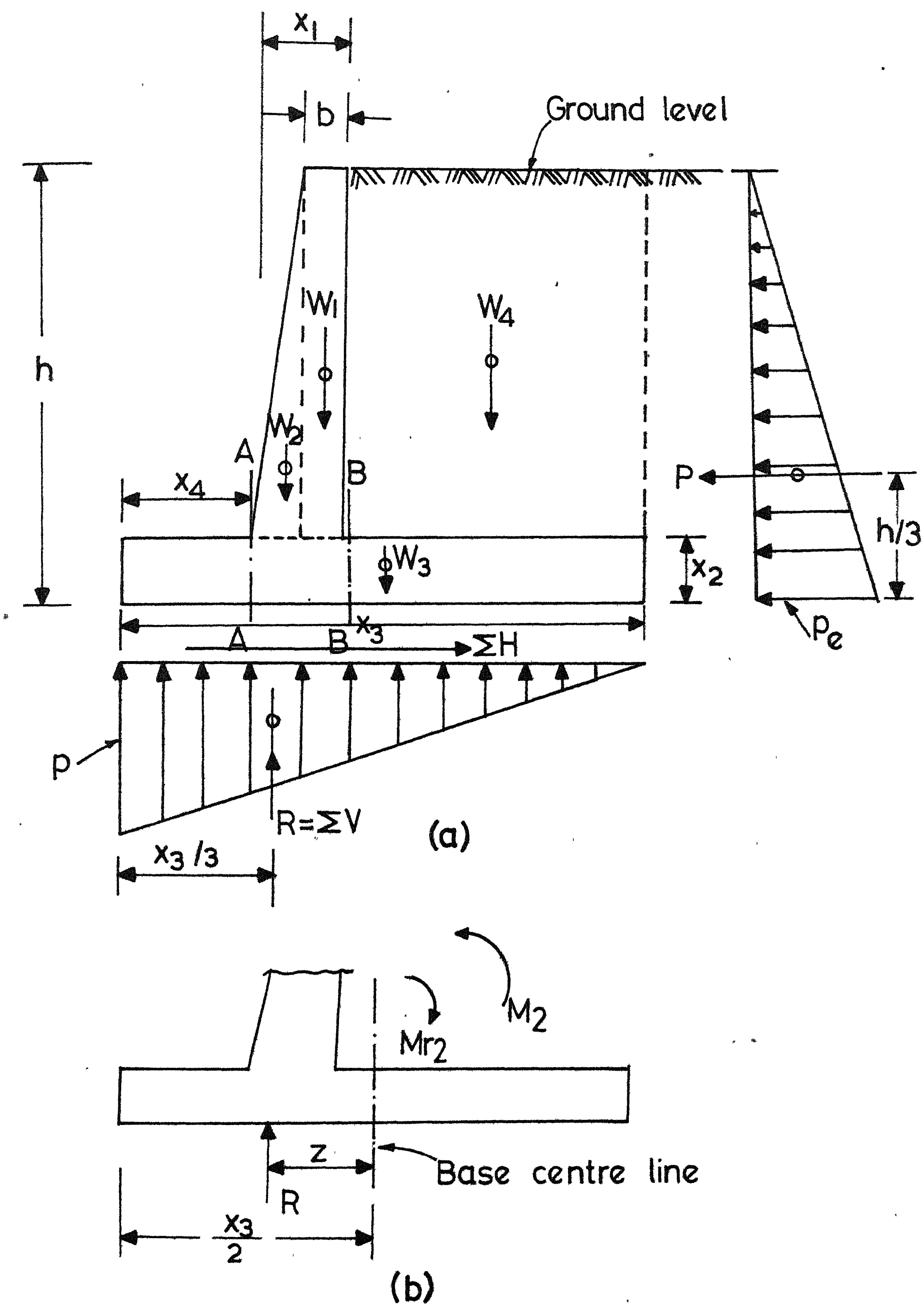


Fig. 5.3 Analysis of a cantilever retaining wall.

(ii) The backfill is assumed to be dry and having a horizontal top level; in other words, no surcharge is assumed in the problem.

(iii) The cantilever retaining wall is of reinforced concrete whose design is governed by the Indian Code of Practice for plain and reinforced concrete, IS : 456 (1964).

(iv) No tension is to be developed in the foundation.

(v) 12 mm diameter plain, mild steel bars are provided for the main reinforcement in the stem and toe slab and 16 mm diameter bars in the heel slab. The distribution steel is of 10 mm diameter bars. Nominal temperature reinforcement is provided in the wall.

The preassigned parameters in the optimization problem are the top width b and the overall height h of the wall [Figure 5.3(a)]. The design variables are: the stem thickness at the bottom, x_1; the thickness of the base slab, x_2; the total base width, x_3 and the toe projection, x_4 so that the design vector $\bar{X}$ is expressible as

$$\bar{X} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} \qquad \ldots 5.5$$

Recalling Article 5.4, the various forces that would form a critical set in the analysis and subsequent design of the present optimization problem, are shown in

Figure 5.3(a). It can be easily noticed that for a 'no-tension-condition' in the foundation, the critical pressure distribution on the base, assumed to be linear, has to be triangular in shape with the maximum ordinate value 'p' at the toe.

5.9 CONSTRAINTS IN THE PROBLEM

The present optimization problem is governed by eighteen constraints, ten of which are the behaviour constraints and remaining eight, the side constraints.

Expressing the design variables $x_1, \ldots, x_4$ and the preassigned parameters b and h in metres, the forces and moments are evaluated as kg and kg-m respectively and per unit length of the wall, taken as one metre. The areas and volumes are in m^2 and m^3 respectively, unless otherwise mentioned.

Following quantities can be evaluated first:

the cross-sectional area $$A = \frac{1}{2}(b + x_1)(h - x_2) + x_2 x_3 \qquad \ldots 5.6$$

soil parameter $$K = \frac{1 - \sin\phi}{1 + \sin\phi} \qquad \ldots 5.7$$

ϕ being the angle of internal friction for the soil retained.

The resultant lateral earth pressure over height h

$$P = \frac{1}{2} w_s . h^2 K \qquad \ldots 5.8$$

acting at a height $\frac{h}{3}$ from the base, where w_s is the unit weight of soil retained.

5.9.1 Behaviour Constraints

5.9.1.1 Minimum stem thickness at the bottom to resist shearing force

The external horizontal shearing force on the stem bottom section

$$F_1 = \frac{1}{2} K \cdot w_s (h - x_2)^2$$

The internal resisting force in reinforced concrete

$$F_{r1} = j(x_1 - c_s) \tau_c$$

where j is a constant associated with the permissible stresses in the concrete and steel used, c_s is the cover provided for main steel in the stem and τ_c the permissible shear stress in concrete. Hence

$$g_1(\bar{X}) = \frac{F_1}{F_{r1}} - 1 \qquad \ldots 5.9(a)$$

5.9.1.2 Minimum stem thickness at the bottom to resist bending moment

External bending moment at the stem bottom section

$$M_1 = \frac{F_1}{3} (h - x_2)$$

The internal moment of resistance for a balanced section is

$$M_{r1} = Q\,(x_1 - c_s)^2$$

where Q is a constant associated with the permissible stresses in the concrete and steel used. So

$$g_2(\bar{X}) = \frac{M_1}{M_{r1}} - 1 \qquad \text{... 5.9(b)}$$

5.9.1.3 Safety against overturning about the toe

Subdividing the cross-section into rectangles and triangles [Figure 5.3(a)], the weights of different portions are

$$W_1 = w_c\, b(h - x_2)\ .$$

$$W_2 = \frac{1}{2} w_c (x_1 - b)(h - x_2)$$

$$W_3 = w_c\, x_3\, x_2\ .$$

w_c being the unit weight of reinforced concrete.

The weight of soil over the heel portion is

$$W_4 = w_s (h - x_2)(x_3 - x_4 - x_1)\ .$$

At the time of overturning, the resultant base pressure passes through the toe and therefore has no moment about the toe. The overturning moment is due to P only and is given by

$$M_2 = \frac{1}{3}\, Ph\ .$$

The balancing moment by the other forces is

$$M_{r2} = W_1(x_4 + x_1 - \frac{b}{2}) + W_2 [x_4 + \frac{2}{3}(x_1 - b)] + \frac{W_3}{2} x_3$$
$$+ \frac{W_4}{2} (x_3 - \frac{x_3 - x_1 - x_4}{2})$$

If 'FS_o' is the factor of safety against overturning,

$$g_3(\bar{X}) = \frac{(FS_o).M_2}{M_{r2}} - 1 \qquad \ldots 5.9(c)$$

5.9.1.4 Safety against sliding

Let $W = W_1 + W_2 + W_3 + W_4$

The external sliding force is $F_2 = P$.

The frictional resisting force by virtue of Equation 5.1 is

$$F_{r2} = c \cdot x_3 + W \tan\phi$$

If 'FS_s' is the factor of safety against sliding, then

$$g_4(\bar{X}) = \frac{(FS_s)F_2}{F_{r2}} - 1 \qquad \ldots 5.9(d)$$

5.9.1.5 No-tension condition in the foundation

If R is the resultant foundation pressure on the base, then

$$R = \frac{p}{2} \cdot x_3$$

acting through the centroid of the pressure triangle as shown in Figure 5.3(a). Considering the equilibrium of the given set of forces, one can note that R = W, and the eccentricity Z of the resultant R with respect to the base centre can be expressed by the equation

$$M_{r2} = M_2 + R\left(\frac{x_3}{2} - Z\right) \quad \text{which yields}$$

$$Z = \frac{x_3}{2} - \frac{M_{r2} - M_2}{W}$$

In the 'no-tension condition', Z is limited to a value $\frac{x_3}{6}$ as seen from Figure 5.3(b) and so

$$g_5(\bar{X}) = \frac{Z}{x_3/6} - 1 \qquad \ldots\ 5.9(e)$$

5.9.1.6 Maximum pressure on the foundation

Referring to Article 5.9.1.5, the maximum pressure p at the toe is given by

$$p = \frac{2R}{x_3} = \frac{2W}{x_3}$$

If p_{per} is the permissible bearing capacity of the foundation, then

$$g_6(\bar{X}) = \frac{p}{p_{per}} - 1 \qquad \ldots\ 5.9(f)$$

5.9.1.7 Minimum thickness of the toe slab to resist shearing force

The external shearing force at section A-A [Figure 5.3(a)] is

$$F_3 = \frac{px_4}{2}\left(1 + \frac{x_3 - x_4}{x_3}\right) - w_c x_4 x_2$$

The internal resisting force in reinforced concrete is

$$F_{r3} = j(x_2 - c_b)\,\tau_c$$

where c_b is the cover provided for the main steel in the base slab. Therefore

$$g_7(\bar{X}) = \frac{F_3}{F_{r3}} - 1 \qquad \ldots\ 5.9(g)$$

5.9.1.8 Minimum thickness of the toe slab to resist bending moment

The external bending moment at section A-A [Figure 5.3(a)] is

$$M_3 = \left[p\,\frac{(x_3 - x_4)}{x_3} \cdot \frac{x_4^2}{2} + \frac{p}{3} \cdot \frac{x_4^3}{x_3}\right] - \frac{w_c}{2}\,x_4^2 x_2$$

The internal moment of resistance of the balanced section is

$$M_{r3} = Q(x_2 - c_b)^2$$

Hence,

$$g_8(\bar{X}) = \frac{M_3}{M_{r3}} - 1 \qquad \ldots\ 5.9(h)$$

5.9.2 Side Constraints

5.9.2.1 Lower and upper bounds on the design variables

Using subscripts 'min' and 'max' to denote the prescribed minimum and maximum values of variables $x_1, \ldots, x_4$ the constraint relations giving the lower and the upper bounds on the design variables are

$$g_{11}(\bar{X}) = \frac{x_{1min}}{x_1} - 1 \qquad \ldots 5.9(k)$$

$$g_{12}(\bar{X}) = \frac{x_{2min}}{x_2} - 1 \qquad \ldots 5.9(l)$$

$$g_{13}(\bar{X}) = \frac{x_{3min}}{x_3} - 1 \qquad \ldots 5.9(m)$$

$$g_{14}(\bar{X}) = \frac{x_{4min}}{x_4} - 1 \qquad \ldots 5.9(n)$$

$$g_{15}(\bar{X}) = \frac{x_1}{x_{1max}} - 1 \qquad \ldots 5.9(p)$$

$$g_{16}(\bar{X}) = \frac{x_2}{x_{2max}} - 1 \qquad \ldots 5.9(q)$$

$$g_{17}(\bar{X}) = \frac{x_3}{x_{3max}} - 1 \qquad \ldots 5.9(r)$$

$$g_{18}(\bar{X}) = \frac{x_4}{x_{4max}} - 1 \qquad \ldots 5.9(s)$$

5.10 VOLUME OF STEEL PER METRE LENGTH OF THE WALL

5.10.1 Stem

Main steel required in the stem portion,

$$A_{s1} = \frac{M_1}{\sigma_{st}\ j(x_1 - c_s)} \cdot 10^4\ \text{cm}^2 \qquad \ldots\ 5.10(a)$$

where σ_{st} is the permissible stress in tension for steel.

As per the Clause 9.2.1 of IS : 456 (1964), the minimum steel should be 0.15% of the gross-sectional area of concrete. So

$$A_{s1} = \frac{0.15}{100} \cdot x_1 \cdot 10^4\ \text{cm}^2 \qquad \ldots\ 5.10(b)$$

The steel required is the *greater* of the two values given by Equations 5.10(a) and 5.10(b).

As 12 mm diameter bars are used (sectional area 1.13 cm^2), the spacing required is $= \frac{100}{A_{s1}} \cdot (1.13)$ cm which will be suitably rounded up to a value, say y_1 and A_{s1} now represents the steel *actually* provided.

Assume that the main steel in the stem is suitably *curtailed* such that, for every four bars, two alternate bars are stopped at a height $\frac{1}{3}\,h_1$, one bar at a height $\frac{2}{3}\,h_1$ and the remaining bar taken for the full height h_1, where $h_1 = h - x_1$.

The volume of main steel in the stem can be written approximately as

. The volume of steel in the toe and the heel portions can be similarly worked out and is only indicated in the following articles.

5.10.2 Toe Slab

$$\text{Main steel, } A_{s3} = \frac{M_3}{\sigma_{st}\ j(x_2 - c_b)} \cdot 10^4\ cm^2 \qquad \ldots 5.16(a)$$

$$\text{or} \qquad = \frac{0.15}{100} \cdot x_2 \cdot 10^4\ cm^2 \qquad \ldots 5.16(b)$$

whichever is greater.

The spacing of 12 mm diameter bars provided (sectional area 1.13 cm^2) is $= \frac{100}{A_{s3}}$ (1.13), and after suitably rounding it to a value y_3, the actual area provided is called A_{s3}

$$\text{Distribution steel, } A_{s4} = \frac{0.15}{100} \cdot x_2 \cdot 10^2\ cm^2 \qquad \ldots 5.17$$

The spacing of 10 mm diameter bars provided is $= \frac{100}{A_{s4}}$ (0.785) and after suitably rounding it to a value y_4, the actual area provided is called A_{s4}.

$$\text{Volume of main steel, } V_{sm2} = A_{s3} \cdot 10^2\ cm^3 \qquad \ldots 5.18$$

$$\text{Volume of distribution steel, } V_{sd2} = A_{s4} \cdot 10^2\ cm^3 \qquad \ldots 5.19$$

Therefore,

$$\text{Volume of steel in toe slab, } V_{st} = V_{sm2} + V_{sd2} \qquad \ldots 5.20$$

5.10.3 Heel Slab

$$\text{Main steel, } A_{s5} = \frac{M_4}{\sigma_{st}\ j(x_2 - c_b)} \cdot 10^4 \text{ cm}^2 \qquad \ldots 5.21(a)$$

$$\text{or} \qquad = \frac{0.15}{100} \cdot x_2 \cdot 10^2 \text{ cm}^2 \qquad \ldots 5.21(b)$$

whichever is greater.

The spacing of 16 mm diameter bars provided (sectional area 2.01 cm^2) is $= \frac{100}{A_{s5}}$ (2.01) and after suitably rounding, it to a value y_5, the actual area provided is called A_{s5}.

Distribution steel, A_{s6} is the same as A_{s4}, provided at spacing y_4.

$$\text{Volume of main steel, } V_{sm3} = A_{s5} \cdot 10^2 \text{ cm}^3 \qquad \ldots 5.22$$

$$\text{Volume of distribution steel, } V_{sd3} = A_{s6} \cdot 10^2 \text{ cm}^3 \qquad \ldots 5.23$$

Therefore,

$$\text{Volume of steel in heel slab, } V_{sh} = V_{sm3} + V_{sd3} \qquad \ldots 5.24$$

If w_{st} is the density of steel in kg/cm^3, the total weight W_{st} of steel may be obtained after increasing it by 10%, to account for bar bending, overlapping, hooks etc. Thus

$$W_{st} = 1.1(V_{ss} + V_{st} + V_{sh})\ w_{st} \qquad \ldots 5.25$$

Also, volume of concrete per running metre of the wall may be obtained on the basis of *gross sectional area* from Equation 5.6 as

$$V_c = A \qquad \ldots 5.26$$

5.12.3 Properties of Concrete and Steel

The concrete mix used for the retaining wall is of grade M-150 and the reinforcement is of plain mild steel bars. Referring to Tables VI and VII of IS : 456 (1964), the permissible values of stresses for this concrete grade and steel are: $\sigma_{cb} = 50$ kg/cm^2, $\tau_c = 5$ kg/cm^2 and $\sigma_{st} = 1400$ kg/cm^2. As per the Clause 6.3.2-(a) of the above reference, the modular ratio 'm' has value 19, so that the design constants are: $j = 0.865$ and $Q = 8.737$ kg/cm^2. The unit weight of reinforced concrete is taken as $w_c = 2400$ kg/m^3.

These values, *in appropriate units*, have been used in Equations 5.9, 5.10(a), 5.16(a) and 5.21(a).

5.12.4 Steel Cover

The cover provided for the main reinforcement in the stem is $c_s = 4$ cm and that in the base slab, $c_b = 5$ cm.

5.12.5 Factors of Safety

The factors of safety against overturning (FS_o) and sliding (FS_s) have been taken as 1.5.

5.12.6 Unit Costs of Steel and Concrete

The material costs of steel and concrete have been taken as: r_s = Rs. 3 per kg and r_c = Rs. 300 per m^3 respectively.

5.12.7 Lower and Upper Bounds on the Design Variables

As suggested by Huntington (1967), the minimum batter for the face should be 1 in 24. On this basis, the value of the lower bound x_{1min} is taken as $(b + \frac{h}{24})$. Based on the section requirements, as suggested by Huntington (1967), Fergusson (1965) and Tschebotarioff (1951), the values of the remaining lower bounds have been fixed as: $x_{2min} = 0.30$ m, $x_{3min} = \frac{h}{2}$ and $x_4 = \frac{h}{6}$.

Similarly, the values of the upper bounds on the design variables have been decided as: $x_{1max} = \frac{h}{6}$, $x_{2max} = \frac{h}{6}$, $x_{3max} = h$ and $x_{4max} = \frac{h}{4}$.

5.13 OPTIMIZATION PROBLEM

Thus the present optimization problem is of constrained type, in the form:

to find

$$\bar{X} = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_4 \end{Bmatrix} \qquad \ldots 5.5$$

so as to minimize the objective function

$$f(\bar{X}) = r_s W_{st} + r_c V_c \qquad \ldots 5.27$$

and subject to the inequality constraints

$$g_j(\bar{X}) \leq 0, \qquad j = 1, 2, \ldots, 18 \qquad \ldots 5.9$$

The solution is attempted by the interior penalty function method which has been explained in Article 3.8. In this case, while obtaining the unconstrained minimum of the penalty function, the method suggested by Powell (1964) has been used, as it needs evaluation of the function values only.

The iterative procedure of the Powell's method can be stated by the following algorithm:-

(i) Choose a starting point $\bar{X}_1$. Set the direction vectors $\bar{S}_i$ equal to the co-ordinate unit vectors, $i = 1,2,\ldots,n$ such that

$$\bar{S}_1 = \begin{Bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{Bmatrix}, \quad \bar{S}_2 = \begin{Bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{Bmatrix}, \quad \ldots, \quad \bar{S}_n = \begin{Bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{Bmatrix}.$$

(ii) Find the minimizing step lengths λ_i^*, $i = 1, 2, \ldots, n$ so that $f(\bar{X}_i + \lambda_i^* \bar{S}_i)$ is a minimum along the direction $\bar{S}_i$, and define

$$\bar{X}_{i+1} = \bar{X}_i + \lambda_i^* \bar{S}_i$$

In minimizing along a direction $\bar{S}_i$, the method of quadratic interpolation can be preferably used, as it needs the function values only.

(iii) Find the integer m, $1 \leq m \leq n+1$ so that the quantity

$$\Delta = f(\bar{X}_{m-1}) - f(\bar{X}_m)$$

is maximum.

(iv) Check for the convergence; if satisfied, terminate the process otherwise go to the next step.

(v) Calculate $f_3 = f(2\bar{X}_{n+1} - \bar{X}_1)$

$$f_1 = f(\bar{X}_1)$$

and, $f_2 = f(\bar{X}_{n+1})$.

(vi) If either $f_3 \geq f_1$ and/or

$$(f_1 - 2f_2 + f_3)(f_1 - f_2 - \Delta)^2 \geq \frac{1}{2}\Delta\,(f_1 - f_3)^2$$

use the old set of directions $\bar{S}_1, \bar{S}_2, \ldots, \bar{S}_n$ for the next cycle of minimizations, and use $\bar{X}_{n+1}$ for the next $\bar{X}_1$ and go to step (ii). Otherwise go to the next step.

(vii) Define the new search direction $\bar{S} = \bar{X}_{n+1} - \bar{X}_n$ and calculate λ^* so that $f(\bar{X}_{n+1} + \lambda^* \bar{S})$ is a minimum along the direction $\bar{S}$. Then start the next cycle of minimizations by taking $(\bar{X}_{n+1} + \lambda^* \bar{S})$ as the starting point and $\bar{S}_1, \bar{S}_2, \ldots, \bar{S}_{m-1}, \bar{S}_{m+1}, \bar{S}_{m+2}, \ldots, \bar{S}_n, \bar{S}$ as the new set of search directions and go back to Step (ii).

5.14 COMPUTER PROGRAMME

The computer programme prepared under the name OPT1 consists of the following subroutines performing functions as indicated.

(i) MAIN -- Reads the data, performs various functions in the optimization procedure using other subroutines and finally gives the optimal

dimensions of the section, the minimum value of the objective function and the corresponding values of constraints in the problem.

(ii) SUMT1 -- Obtains the constrained minimum of a function using the sequential unconstrained minimization technique (SUMT).

(iii) PENLTY -- Lists the given set of constraints and generates therefrom the objective and the penalty functions at a design point $\bar{X}$.

(iv) POWELL -- Finds the unconstrained minimum of a function by the conjugate directions method.

(v) UNIDIR -- Minimizes a function in one direction by the method of quadratic interpolation.

The results for all the cases above have been discussed and the conclusions drawn, in Chapter 6.

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6. DISCUSSION OF RESULTS AND CONCLUSIONS

6.1 COMPUTATIONAL FEATURES OF THE PROGRAMMES

The computational work is carried out on IBM 7044 and DEC 1090 computer systems at the Indian Institute of Technology, Kanpur and DEC 1077 system at the Tata Institute of Fundamental Research, Bombay.

By using the optimization programmes OPT1, OPT2, OPT3 and OPT4, the optimal solution in every case has been obtained using different starting points. However, during their execution, these programmes exhibit certain features.

6.1.1 Programmes OPT2, OPT3 and OPT4

The starting design point for these programmes is chosen from the available range of rolled steel sections given in SP : 6(1) (1964). Each of such points represents a practically feasible solution of the problem.

The characteristic feature of these programmes is that, the optimum value of the objective function is decreased as the starting point successively approaches the optimal design point. During this process, if one starts from a design point corresponding to a very heavy section of equal flange widths and thicknesses, the optimal section, thus obtained, has unequal flange widths and thicknesses. This is obvious because the programmes OPT2, OPT3 and OPT4

handle each of the optimization problems as a general six-design-variable problem. However, as the starting point approaches the optimum one, it is observed that while the optimal solution is arrived at, the penalty function is minimized with an increase in the value of the objective function. As reported by Fox (1971), such type of behaviour is not uncommon with the present form of the penalty function chosen for the constrained minimization problems. While this is unfortunate, no other course is usually possible.

The requisite logical provision incorporated in the programme switches the final iterative solution back to the starting point, which is a section with equal flange widths and thicknesses. By successively decreasing the sectional dimensions, the starting point can be taken as close to the optimum solution as possible. The optimal solution, thus obtained, is invariably a section with equal flange widths and thicknesses.

A typical case in the programme OPT2, which illustrates all these foregoing features, is given in the form of Table 6.1. It can be seen that the optimal solution is obtained for the starting point no. 8 which is nearest to the optimal point. This is further explained by the fact that the next point no. 9 just falls into the infeasible region.

Table 6.1

Variation of Optimal Solution With Different Starting Points

Span: 3 m ; Boundary Conditions: fixed

Loading: uniformly distributed ; Intensity: 1000 kg/m

Lateral Eccentricity of Loading: 0.1 bf_1 (from Table 6.2)

Sl. no.	Starting Design Point (dimensions in mm)							Optimal Design Point (dimensions in mm)							Remarks
	bf_1	tf_1	bf_2	tf_2	hw	tw	Weight in kg/m	bf_1	tf_1	bf_2	tf_2	hw	tw	Weight in kg/m	
1	125.0	8.7	125.0	8.7	250.0	6.1	29.05	76.6	8.8	74.4	8.9	214.2	4.6	18.22	IS 250
2	100.0	8.6	100.0	8.6	225.0	5.8	23.75	74.3	8.6	72.1	8.7	203.9	4.5	17.14	IS 225
3	100.0	7.3	100.0	7.3	200.0	5.4	19.94	100.0	7.3	100.0	7.3	200.0	5.4	19.94	IS 200
4	90.0	6.9	90.0	6.9	175.0	5.1	16.76	90.0	6.9	90.0	6.9	175.0	5.1	16.76	IS 175
5	80.0	6.8	80.0	6.8	150.0	4.8	14.19	80.0	6.8	80.0	6.8	150.0	4.8	14.19	IS 150
6	75.0	6.5	75.0	6.5	125.0	4.4	11.97	75.0	6.5	75.0	6.5	125.0	4.4	11.97	IS 125
7	70.0	6.5	70.0	6.5	120.0	4.3	11.19	70.0	6.5	70.0	6.5	120.0	4.3	11.19	-
8	69.0	6.5	69.0	6.5	119.0	4.3	11.05	69.0	6.5	69.0	6.5	119.0	4.3	11.05	Optimal solution*
9	68.0	6.47	68.0	6.47	118.0	4.29	10.88	Starting point infeasible; hence no solution.							

*This result is finally entered in Table 6.23.

active constraint is on the normal stress for a span upto 4.0 m, while for higher spans it is on the deflection. The optimum weight increase is from 21.29 to 50.00 kg/m for a loading intensity of 3000 kg/m, the active constraint in each case being that on the normal stress.

For a cantilever, the optimum weight range is from 8.30-18.88 kg/m and 11.34-25.44 kg/m for the loading intensities of 1000 and 2000 kg/m respectively, the span ranging from 1.0 to 2.0 m. In each case, the constraint on the deflection remains active. For a loading intensity of 3000 kg/m, the constraint on the normal stress is seen to be active except for a span of 2.0 m where the constraints on the normal stress and the deflection are both active (Table 6.7). The optimum weight varies from 13.45 to 30.52 kg/m in this case.

6.2.1.2 Linearly varying load

Tables 6.8 through 6.19 show the results in this case.

In the case of fixed supports, for a span range of 3.0 to 6.0 m, the optimum weight increases from 8.16 to 19.28 kg/m and 15.52 to 36.07 kg/m as the loading intensity ranges change from 0-1000 kg/m to 2000-3000 kg/m respectively. In each case the active constraint happens to be the normal stress in the beam, excepting when the span is 3.0 m and the loading intensity 1000 kg/m. In this case,the prescribed

lower bounds on the top and the bottom flange widths act as the active constraints, as indicated in Table 6.8.

When the supports are simple, for a span increase from 3.0 to 6.0 m, the variation in the optimum weight is from 9.66 to 21.74 kg/m and 19.17 to 44.56 kg/m, as the loading intensity range changes from 0-1000 kg/m to 2000-3000 kg/m respectively. For loading intensities of 0-1000 kg/m, 0-2000 kg/m, 0-3000 kg/m and 1000-2000 kg/m, the constraint on the deflection is seen to be active, excepting for a span of 6.0 m (loading intensity being 0-1000 kg/m) and a span of 5.0 m (loading intensity being 0-2000 kg/m) where the constraints on the stress and deflection are both active, as indicated in Tables 6.8 and 6.10 respectively. When the loading intensity is 1000-3000 kg/m, the constraint on the stress is active upto a span of 4.0 m after which the constraint on the deflection is active (Table 6.16). Lastly, for the loading intensity of 2000-3000 kg/m, the constraint on the stress is found to be active for all spans, except when it is 6.0 m in which case the constraints on the stress and on the deflection are both found to be active (Table 6.18).

In the case of a cantilever, for a loading intensity of 1000-0 kg/m, the lower bounds on the top and bottom flange widths are seen to be the active constraints for spans of 1.0 and 1.5 m (Table 6.9). This pair of active constraints is also observed when the span is 1.0 m and loading intensities are 2000-0 kg/m (Table 6.11) and

3000-0 kg/m (Table 6.13) respectively. The variation of the optimum weight is from 8.16 to 11.66 kg/m and 11.97 to 27.12 kg/m as the loading intensity changes from 1000-0 kg/m to 3000-2000 kg/m. For a span of 1.5 m and loading intensity 3000-2000 kg/m, the constraints on the stress and the deflection are active as seen from Table 6.19. In the remaining cases, except those mentioned above, the constraint on the deflection is active.

6.2.1.3 Nodal loads

The results are presented in Tables 6.20 through 6.22.

For fixed supports, the variation in the optimum weight is from 8.16 to 21.74 kg/m and from 14.39 to 33.87 kg/m for a span increase of 3.0 to 6.0 m, when the equivalent intensity of the nodal loads changes from 1000 to 3000 kg/m.

The corresponding figures for the optimum weight variation in the case of simple supports are 11.97 to 28.84 kg/m and 21.29 to 50.00 kg/m respectively.

For a span of 3.0 m under the fixed support conditions, when the equivalent loading intensity is 1000 kg/m, the optimal solution is governed by the constraints on the lower bounds on the top and the bottom flange widths. On the other hand, the constraint on the deflection is active when the span is 4.0 m under the simple support conditions. In all the rest of the cases,the constraint on the normal stress is active.

6.2.1.4 Uniformly distributed load with a lateral eccentricity

Tables 6.23 through 6.37 give the results for this case.

In the case of fixed supports, for a loading intensity of 1000 kg/m at a lateral eccentricity of 0.1 bf_1 (bf_1 being the top flange width of the *optimal* section for the corresponding case of the in-plane loading), the optimum weight increases from 11.05 to 28.84 kg/m as the span changes from 3.0 to 6.0 m. When the lateral eccentricity is increased to values 0.2 bf_1 and 0.3 bf_1, the corresponding ranges of the optimum weight are 12.57-31.82 kg/m and 14.11-34.76 kg/m respectively.

For these values of the lateral eccentricity, the corresponding increases in the ranges of the optimum weight are 17.13-39.60 kg/m, 19.94-47.19 kg/m, 22.97-54.39 kg/m and 22.05-51.49 kg/m, 26.06-62.61 kg/m and 29.04-71.67 kg/m when the loading intensity assumes values of 2000 kg/m and 3000 kg/m respectively.

In all these cases, the constraint on the normal stress is seen to be active.

However, in the case of simple supports, when the loading intensity is 1000 kg/m at a lateral eccentricity of 0.1 bf_1, the increase in the optimum weight is from 12.84 to 31.19 kg/m as the span changes from 3.0 to 6.0 m. For spans upto 4.0 m, the constraint on the deflection is active

while for higher span values, the constraint on the normal stress takes over as the active one, as can be seen from Table 6.23. *Both the normal stress and the deflection constraints are simultaneously active for spans ranging from 4.0 to 4.5 m approximately.*

For the lateral eccentricity values of 0.2 bf_1 and 0.3 bf_1, the corresponding increases in the optimum weight are from 13.56 to 33.70 kg/m and from 14.80 to 36.07 kg/m respectively.

As the loading intensity is increased to 2000 and 3000 kg/m, the respective ranges for these three values of the eccentricity are 19.68-46.13 kg/m, 22.34-52.25 kg/m, 25.01-58.87 kg/m and 25.64-59.35 kg/m, 29.85-68.67 kg/m and 33.50-79.29 kg/m. In all these cases, the constraint on the normal stress is the active one.

In the case of a cantilever, when the loading intensity is 1000 kg/m at an eccentricity of 0.1 bf_1, the optimum weight varies from 8.31 to 19.17 kg/m as the span changes from 1.0 to 2.0 m. This variation for the loading eccentricity of 0.2 bf_1 is from 9.36 to 22.34 kg/m.

For the increase in loading to the values of 2000 kg/m and 3000 kg/m, the corresponding variation in the optimum weight for these two values of the lateral eccentricity are 13.11-29.55 kg/m, 15.11-34.21 kg/m and 16.76-38.55 kg/m and 19.40-45.40 kg/m respectively.

In all these cases, the constraint on the normal stress is found to be active.

6.2.2 Optimization of Thin-Walled Sections under Stability Constraints

The results of this optimization problem are shown graphically in Figure 6.1.

It may be noticed that for all types of boundary conditions, the optimum weight of the column increases non-linearly with the increase in the height of the column. For fixed end conditions, the optimum weight increases from 3.16 to 24.16 kg/m as the column height increases from 3.0 to 6.0 m. For a column height of 3.0 m, the lower bounds on the top and the bottom flange widths act as the active constraints. For the remaining column heights, the constraint on the maximum slenderness ratio of the column is active.

When the ends are simply supported, the variation in the optimum weight is from 11.50 to 26.50 kg/m as the column height increases from 3.0 to 4.5 m. The corresponding figures for a cantilever column are 24.40 to 32.83 kg/m for a height increase from 3.0 to 4.0 m. In all these cases, the constraint on the maximum slenderness ratio of the column is seen to be active.

6.2.3 Optimization of Thin-Walled Sections under Dynamic Constraints

The results for this optimization problem are presented graphically in Figures 6.2 through 6.4.

For fixed supports, the optimum weight increases from 8.16 to 10.88 kg/m, as the span changes from 3.0 to 6.0 m and the upper and lower bounds on the first value of transverse vibrational frequency have a broad range. Upto the span of 5.0 m, the active constraints are the lower bounds on the top and the bottom flange widths of the section. For the remaining spans, the lower bound on the first value of natural frequency is seen to be active.

The corresponding variation in the optimum weight is from 8.16 to 33.50 kg/m when the bounds on the first frequency have a narrow range. In this case, the lower bounds on flange widths govern the optimal design for a span upto 3.50 m, after which the lower bound on the first frequency behaves as the active constraint.

When the supports are fixed, for the increase in the span from 3.0 to 6.0 m, the optimum weight increases from 8.16 to 19.03 kg/m and from 9.50 to 41.28 kg/m when the range of frequency bounds is broad and narrow respectively. In the case of a broad range, the lower bounds on flange widths govern the optimal design for spans upto 4.0 m. Otherwise, the active constraint is always the lower bound on the first natural frequency.

In the case of a cantilever, for a variation of span from 1.0 to 2.0 m, the optimum weight enhances from 8.16 to 37.84 kg/m and from 27.55 to 48.63 kg/m when the

bounds on the first natural frequency have a broad and a narrow range respectively. In the case of a broad range, when the span is 1.0 m, the lower bounds on the two flange widths form the active constraints while in the remaining cases the lower bound on the first natural frequency always governs the optimal design.

6.2.4 Optimization of a Cantilever Retaining Wall

The results for this optimization problem are presented graphically in Figures 6.5 through 6.7.

As can be naturally expected, for any type of soil, the optimal cost rises with the increase in height and top width. However, the salient features of results are discussed according to the type of soil used for the backfill.

6.2.4.1 Gravel

For a top width of 0.20 m and the wall height upto 3.5 m, the optimal cost per running metre of the wall varies from Rs. 454.44 to Rs. 578.13. The constraint governing the optimal design is the lower bound on the stem root thickness. However, for a height of 4.0 m, the active constraint changes to the lower bound on the base slab thickness the corresponding optimum cost being Rs. 696.16. A further increase in the wall height from 4.5 to 6.0 m enhances the optimum cost from Rs. 866.94 to Rs. 1530.29 and the active constraint is the lower bound on the base slab width.

6.2.4.5 Clay

For a top width of 0.20 m, the increase in the optimum cost is from Rs. 518.33 to Rs. 2012.39 as the wall height increases from 3.0 to 6.0 m. For a height upto 3.50 m, the lower bound on the stem root thickness governs the design. When it is above 4.0 m, the active constraint is either the lower bound on stem root thickness to resist bending moment or the no tension condition in the foundation. Similar trend of the active constraints is observed when the top width has values of 0.25 and 0.30 m. In these cases, as the height increases from 3.0 to 6.0 m, the optimum cost increase is from Rs. 562.49-2029.66 and Rs. 601.92-2073.37 respectively.

6.3 CONCLUSIONS

The conclusions are drawn separately for every optimization problem handled so far.

6.3.1 Optimization of Thin-Walled Sections under Static Constraints

This problem has been solved for four different cases of the loading and as such it is found logical to draw conclusions for each loading case and then deduce overall conclusions.

6.3.1.1 Uniformly distributed load

As the loading intensity varies from 1000 to 3000 kg/m, the optimum weight increase in case of fixed support

conditions is 86.40% for a span of 3.0 m, but it reduces to 55.02% for a span of 6.0 m. The corresponding values in respect of simple supports and a cantilever are 65.80% - 68.35% and 62.05% -61.65%, for a span variation of 3.0-6.0 m and 1.0-2.0 m respectively.

The effect of boundary conditions may also be noted with interest. For a span of 3.0 m, as the supports change from fixed to simple, the increase in optimum weight falls from 41.87% to 26.20% when the loading intensity increases from 1000 to 3000 kg/m. However, for a span of 6.0 m the increase in optimum weight has a rise from 18.75% to 28.96%.

Hence, for smaller load intensities, fixed beams are very much economical than simple beams, provided the spans are small. For higher intensities also, significant economy can be achieved by using fixed beams for small or large spans.

It is worthwhile to note the variation in the type of active constraint in the optimal design. For fixed supports, the constraint on the normal stress is active under all spans and loading intensities. In case of simple supports, the deflection constraint governs for small loading intensities. For medium loading intensities, the stress constraint is active for smaller spans, and the deflection constraint for larger spans. When the loading intensities are large, the stress constraint dictates the optimal design for all spans.

In case of cantilever beams, for all spans, the deflection constraint is governing for small and medium loading intensities but for larger intensities, the stress constraint dominates.

6.3.1.2 Linearly varying load

Generally speaking, for this type of loading, the constraint on the normal stress remains active for all spans and loading intensity ranges, when the supports are fixed. In the case of simple supports, the deflection constraint governs the optimal design for all spans when the ranges of loading intensities are small or medium. If these are large, the stress constraint is active for small spans while the deflection constraint dominates in case of large spans. In the case of a cantilever beam, the deflection constraint is active under all spans and loading intensity ranges.

In the case of fixed supports, for an increase in loading intensity range from 0-1000 kg/m (average value 500 kg/m) to 2000-3000 kg/m (average value 2000 kg/m), the optimum weight enhances by 90.19% and 87.09% for spans of 3.0 m and 6.0 m respectively. Corresponding values in case of simple supports and a cantilever are 98.44%-104.96% and 46.69%-132.59%, when the span variation in the respective case is 3.0-6.0 m and 1.0-2.0 m.

This clearly shows that with the increase in loading intensity, cantilever beams in respect of the

optimum weight are very much sensitive to the increase in span, than the fixed or simply supported beams. Hence, for a linearly varying load, it is desirable to cut down the spans of cantilever beams to reduce the structural cost, ultimately.

One may also investigate the possibility for a choice between simple and fixed beams for equal spans. In the former type of beam, for a range of loading intensity from 0-1000 kg/m, the increase in the optimum weight is 18.38%, when the supports are made simple for a span of 3.0 m. Similar increase for a span of 6.0 m is 12.76%. Corresponding values for a loading intensity range of 2000-3000 kg/m are 23.52% and 23.54% respectively.

This clearly suggests that for larger load intensity ranges, fixed beams are economical than simple beams for smaller or larger spans. For smaller load intensity ranges, fixed beams with smaller spans may be somewhat economical; but for large spans the economy achieved is only marginal.

6.3.1.3 Nodal loads

The constraint on the normal stress is always active excepting the case of simple supports under small spans and loads, when the deflection constraint prevails.

When the nodal loads have equivalent intensity of 1000 kg/m, for spans of 3.0 and 6.0 m, the increase in the

optimum weight by changing the support conditions from fixed to simple, is 46.69% and 32.66% respectively. Corresponding figures are 47.95% and 47.62% respectively, when the equivalent intensity has a value of 3000 kg/m.

In practice, nodal loads can assume the form of reactions transferred on a main girder, from the cross-girder systems. When the load magnitudes are higher, the fixed beams are expected to be economical under all spans. Even for smaller load magnitudes, the economy achieved is comparatively more for smaller spans.

6.3.1.4 Uniformly distributed load with a lateral eccentricity

Under all loading intensities with different eccentricities, the constraint on the normal stress is seen to govern the optimal design, for almost all spans and boundary conditions; except, in the case of simple beams carrying loads of smaller intensities and eccentricities, the deflection constraint is active, when the spans are smaller.

When the loading intensity is 1000 kg/m, for a smaller eccentricity of 0.1 bf_1, the increase in the optimum weight due to the change-over of suppprt conditions from simple to fixed, is 16.20% and 8.15% respectively as the span varies from 3.0 to 6.0 m. For a large eccentricity of 0.3 bf_1, these increases are 4.89% and 3.77% respectively.

Hence, for smaller loading intensities, fixed beams will be economical than simple beams, if the loading eccentricity and the span have smaller values.

For a larger loading intensity of 3000 kg/m, with the increase in span from 3.0 to 6.0 m, corresponding variation in the optimum weight for the loading eccentricity value of 0.1 bf_1 is 16.28%-15.27%; for 0.3 bf_1, it is 15.36%-10.63%. Hence, for larger loading intensities having smaller eccentricities, fixed beams offer more economy than simple beams, for all spans; for larger eccentricities, significant economy can be expected for smaller spans.

In the case of cantilever beams, for all loading intensities and spans, the increase in the optimum weight is rather insignificant, being of the order of 15% when the loading eccentricity is changed from 0.1-0.2 bf_1. Therefore, if the situation demands the necessity of larger eccentricites, these are permissible, without greatly affecting the economy.

As the last investigation in this type of loading, values of the optimum weight are compared with those obtained for the in-plane loading under identical spans and loading intensities. From this information, one can have an idea of the significance of lateral eccentricities in the structural design, under various boundary conditions.

For fixed beams, under a small loading intensity with an eccentricity of 0.1 bf_1, the increase in the optimum weight is around 15-20%; for a large eccentricity

of 0.3 bf_1, the increase is 40-50%. For large loadings, such increases for small and large eccentricities are about 30% and 75% respectively. Hence barring, perhaps, the small intensity loadings with smaller eccentricities, the laterally eccentric loads should be scrupulously considered in the design of fixed beams.

In simple beams, such figures are around 5% and 20% for small loads and, 15% and 60% for large loads, when the eccentricities are small and large respectively. Hence, in the design of simple beams, only large eccentricities become significant in the design, for all intensities of loadings.

In cantilevers, for small and large eccentricities, similar values are around 5% and 15%, and 25% and 50% respectively. This signifies that large loadings, even at a small eccentricity, need special consideration in the design of cantilevers.

6.3.1.5 General remarks on optimal sections under static constraints

Generally speaking, for any type of loading described above, the fixed beams are economical for all spans, particularly under large loading intensities; for small intensities, they are economical for small spans only.

In the design of gridworks, the joints need not always be designed as rigid, thus imposing the fixed boundary conditions on the grid members. In some loading cases, even

simple boundary conditions could lead to an optimal design of that grid member whose joints are designed for such boundary conditions.

Tables 6.2 through 6.37 provide optimal sectional dimensions for an I-section for various loadings and boundary conditions. Given a flexibility in the production, such members can be rolled and used to economize thin-walled structures, particularly when the component members are required on a large scale. If the situation does not permit this, the nearest available section would provide a practically feasible optimal structural design.

6.3.2 Optimization of Thin-Walled Sections under Stability Constraints

For a column height of 3.0 m, the optimum weight increases by 40.93% as the support conditions change from fixed to simple, thus doubling the effective height; however, a further increase of 112.17% results as it becomes a cantilever column and the effective height is further doubled. Corresponding values for a column height of 4.0 m are 129.94% and 37.02% respectively.

Hence, generally speaking, for small heights, columns with simple supports may not be so uneconomical with respect to the fixed columns but the cantilever columns would be highly uneconomical as compared to simple ones.

decreased, it is observed that in these soil types, the lower bound on the stem root thickness governs the optimal design for small heights, while for larger heights the constraint on resistance against sliding is active. This is particularly true in case of smaller top widths, which ultimately provide smaller base widths.

The situation is slightly different in the case of silt and clay which have poor bearing capacities but possess cohesive strength that adds to their resistance against sliding. For smaller heights, the lower bound on stem root thickness governs the optimum design while for larger heights the constraint on the safety against sliding is active, this being true for all values of top widths. Comparatively, silt, having higher bearing capacity but lower cohesion, is expected to lead to the optimal design that is economical upto a certain height, than that obtained with clay as the backfill. For heights greater than this, when the constraint on the safety against sliding is bound to govern, optimal design in case of clay, which possesses more cohesion, is bound to be cheaper.

Referring to Figures 6.5 through 6.7, it is deducible that in case of silty backfills, for different top widths, the optimal designs of cantilever retaining walls are economical upto a height of 4.5-4.75 m. After this, height, clayey soils yield a more economical design, although the saving may be marginal, say of the order of 10%.

6.3.5 Computational Algorithms

The computer programme OPT1 deals with the optimization of a cantilever retaining wall, a structure belonging to the thick-walled category. The optimal designs of thin-walled rolled steel sections under the Static, the Stability and the Dynamic Constraints are handled by programmes OPT2, OPT3 and OPT4 respectively. Both these constrained optimization problems are approached using the Interior Penalty Function method; but in each problem, different methods are adopted for the unconstrained minimization involved in the procedure, and these present different features.

In programmes OPT2, OPT3 and OPT4, the Davidon-Fletcher-Powell algorithm is applied which makes use of the gradients of the objective function. Although numerically stable even under eccentric functions, in the present problems it is seen to present difficulties while evaluating gradients at points near the constraints. As can be seen from the optimal solutions presented in Tables 6.2 through 6.37, all optimum design points are bound points. Hence, while using the search technique, as the design point approaches the active constraint, the penalty function tries to 'blow off', which subsequently presents difficulties in evaluating gradients, thus halting the minimization process. The requisite logical provision incorporated in the programmes, however, takes care of such a situation.

The programme OPT1 employs the Powell's method, that incorporates modifications for handling non-quadratic objective functions. This method needs function evaluations only and hence in the vicinity of the optimal solution, normally a bound one, no numerical instability is experienced.

Regarding the computational accuracy, both algorithms are seen to be equally satisfactory.

6.4 SUGGESTIONS FOR FURTHER WORK

The present thesis is just a beginning of the problem of optimization of thin- and thick-walled structures in Civil Engineering practice.

The problem can be expanded to include various other types of loadings, analysed using different techniques and optimized with due consideration to many other varieties of constraints. Thus, the next steps suggested to enhance the scope of the present problem are as follows:

(i) In the optimization of thin-walled sections, the stability and/or dynamic constraints can be included along with the static constraints, so that optimal designs could be compared for different combinations of constraints.

(ii) As the reactions from the cross-girders can, in general, be a force and a twisting and/or bending moment, such concentrated bending and twisting moments can be considered among the nodal loads.

increase falls to an insignificant 8.27% for a wall height of 6.0 m. Similar trend is observed in cases of other soil types considered namely, dense sand (22.4%-4.63%), loose sand (15.89%-1.93%), silt (17.65%-1.67%) and clay (16.12%-3.03%). This clearly shows that for any type of soil retained, the effect of top width on the optimum cost of the cantilever retaining wall is significant at smaller heights only; with larger heights, the effect dies out rapidly.

6.3.4.2 Significance of active constraints

Amongst the five different classes of soil considered, gravel, dense sand and loose sand belong to the cohesionless type while the remaining two, namely silt and clay are cohesive. In the cohesionless variety, gravel possesses high values of the bearing capacity and the angle of internal friction, while these values get successively reduced for dense sand and loose sand. Naturally, in case of gravel, even at higher wall heights, sufficient frictional resistance against sliding is assured by its high value of the angle of internal friction. Hence, the lower bound on the stem root thickness governs the design. However, in case of dense sand and loose sand, the decreasing values of bearing capacity, demand higher base widths and subsequently higher wall dimensions. This obviously results in successive increase in the cost for dense sand and loose sand, it being maximum in the latter case. Again, the frictional resistance getting

Table 6.2

Optimal Sectional Dimensions

Loading; uniformly distributed

Intensity: 1000 kg/m ; Lateral Eccentricity: nil

Span in m	Fixed Supports								Simple Supports							
	Optimal dimensions in mm						Optimum weight in kg/m	Active constraint/s	Optimal dimensions in mm						Optimum weight in kg/m	Active constraint/s
	bf_1	tf_1	bf_2	tf_2	hw	tw			bf_1	tf_1	bf_2	tf_2	hw	tw		
3.0	56.0	6.4	56.0	6.4	106.0	4.1	9.05	g_{13}^{*}	77.0	6.6	77.0	6.6	135.0	4.6	12.84	g_{15}^{**}
4.0	77.2	6.6	77.2	6.6	136.0	4.6	12.91	g_{13}	92.4	7.0	92.4	7.0	181.0	5.2	17.49	g_{15}
5.0	96.0	7.1	96.0	7.1	190.0	5.2	18.52	g_{13}	100.0	8.4	100.0	8.4	221.0	5.7	23.11	g_{13}, g_1
6.0	106.0	8.6	106.0	8.6	231.0	5.9	25.01	g_{13}	127.4	8.7	127.4	8.7	254.0	6.2	29.70	g_{15}

*Maximum normal stress in the beam.

**Maximum deflection in the beam.

Table 6.3

Optimal Sectional Dimensions

Loading: uniformly distributed

Intensity: 1000 kg/m ; Lateral Eccentricity: nil

Span in m	Cantilever							
	Optimal dimensions in mm						Optimum weight in kg/m	Active constraints
	bf_1	tf_1	bf_2	tf_2	hw	tw		
1.0	51.0	6.4	51.0	6.4	101.0	4.0	8.30	g_{15}^{**}
1.5	78.8	6.7	78.8	6.7	144.0	4.7	13.65	g_{15}
2.0	96.8	7.2	96.8	7.2	192.0	5.3	18.88	g_{15}

**Maximum deflection in the beam.

Table 6.4

Optimal Sectional Dimensions

Loading: uniformly distributed

Intensity: 2000 kg/m ; Lateral Eccentricity: nil

Span in m	Fixed Supports								Simple Supports							
	Optimal dimensions in mm						Optimum weight in kg/m	Active constraint/s	Optimal dimensions in mm						Optimum weight in kg/m	Active constraint/s
	bf_1	tf_1	bf_2	tf_2	hw	tw			bf_1	tf_1	bf_2	tf_2	hw	tw		
3.0	78.4	6.7	78.4	6.7	142.0	4.7	13.45	g_{13}^{*}	90.4	6.9	90.4	6.9	176.0	5.1	16.87	g_{13}
4.0	95.6	7.1	95.6	7.1	189.0	5.3	18.52	g_{13}	100.0	8.6	100.0	8.6	225.0	5.3	23.75	g_{13}
5.0	110.0	8.6	110.0	8.6	235.0	5.9	25.81	g_{13}	134.6	8.8	134.6	8.8	266.0	6.3	31.67	g_{15}^{**}
6.0	137.0	8.8	137.0	8.8	270.0	6.3	32.32	g_{13}	156.0	9.6	156.0	9.6	310.0	6.8	40.01	g_{15}

*Maximum normal stress in the beam.

**Maximum deflection in the beam.

Table 6.5

Optimal Sectional Dimensions

Loading: uniformly distributed

Intensity: 2000 kg/m ; Lateral Eccentricity: nil

Span in m	Cantilever							
	Optimal dimensions in mm						Optimum weight in kg/m	Active constra-int/s
	bf_1	tf_1	bf_2	tf_2	hw	tw		
1.0	71.0	6.5	71.0	6.5	121.0	4.3	11.34	g_{15}^{**}
1.5	92.8	7.0	92.8	7.0	182.0	5.2	17.61	g_{15}
2.0	108.0	8.6	108.0	8.6	233.0	5.9	25.44	g_{15}

**Maximum deflection in the beam.

Table 6.8

Optimal Sectional Dimensions

Loading: linearly varying

Intensity: 0-1000 kg/m ; Lateral Eccentricity: nil

Span in m	Fixed Supports								Simple Supports							
	Optimal dimensions in mm						Optimum weight in kg/m	Active constraint/s	Optimal dimensions in mm						Optimum weight in kg/m	Active constraint/s
	bf_1	tf_1	bf_2	tf_2	hw	tw			bf_1	tf_1	bf_2	tf_2	hw	tw		
3.0	50.0	6.4	50.0	6.4	100.0	4.0	8.16	$g_1^{\oplus}, g_3^{\oplus\oplus}$	60.0	6.4	60.0	6.4	110.0	4.2	9.66	g_{15}^{**}
4.0	64.0	6.5	64.0	6.5	114.0	4.2	10.28	g_{13}^{*}	78.8	6.7	78.8	6.7	144.0	4.7	13.65	g_{15}
5.0	81.2	6.8	81.2	6.8	153.0	4.8	14.51	g_{13}	92.0	7.0	92.0	7.0	180.0	5.2	17.37	g_{15}
6.0	98.0	7.2	98.0	7.2	195.0	5.3	19.28	g_{13}	100.0	7.9	100.0	7.9	212.0	5.6	21.74	g_{13}, g_{15}

$\oplus$Lower bound on the top flange width.

$\oplus\oplus$Lower bound on the bottom flange width.

*Maximum normal stress in the beam.

**Maximum deflection in the beam.

Table 6.9

Optimal Sectional Dimensions

Loading: linearly varying

Intensity: 1000-0 kg/m ; Lateral Eccentricity: nil

Span in m	Cantilever							
	Optimal dimensions in mm						Optimum weight in kg/m	Active constraint/s
	bf_1	tf_1	bf_2	tf_2	hw	tw		
1.0	50.0	6.4	50.0	6.4	100.0	4.0	8.16	$g_1^{\oplus}$, $g_3^{\oplus\oplus}$
1.5	50.0	6.4	50.0	6.4	100.0	4.0	8.16	g_1, g_3
2.0	73.0	6.5	73.0	6.5	123.0	4.4	11.66	g_{15}^{**}

⊕Lower bound on the top flange width.

⊕⊕Lower bound on the bottom flange width.

**Maximum deflection in the beam.

Table 6.10

Optimal Sectional Dimensions

Loading: linearly varying

Intensity: 0-2000 kg/m ; Lateral Eccentricity: nil

Span in m	Fixed Supports								Simple Supports							
	Optimal dimensions in mm						Optimum weight in kg/m	Active constraint/s	Optimal dimensions in mm						Optimum weight in kg/m	Active constraint/s
	bf_1	tf_1	bf_2	tf_2	hw	tw			bf_1	tf_1	bf_1	tf_2	hw	tw		
3.0	63.0	6.5	63.0	6.5	113.0	4.2	10.11	g_{13}^{*}	77.0	6.6	77.0	6.6	135.0	4.6	12.84	g_{15}^{**}
4.0	80.8	6.8	80.8	6.8	152.0	4.8	14.39	g_{13}	92.4	7.0	92.4	7.0	181.0	5.2	17.49	g_{15}
5.0	100.0	7.4	100.0	7.4	202.0	5.4	20.23	g_{13}	100.0	8.4	100.0	8.4	221.0	5.7	23.11	g_{13}, g_{15}
6.0	114.0	8.7	114.0	8.7	239.0	6.0	26.70	g_{13}	127.4	8.7	127.4	8.7	254.0	6.2	29.70	g_{15}

*Maximum normal stress in the beam.

**Maximum deflection in the beam.

Table 6.11

Optimal Sectional Dimensions

Loading: linearly varying

Intensity: 2000-0 kg/m ; Lateral Eccentricity: nil

Span in m	Cantilever							
	Optimal dimensions in mm						Optimum weight in kg/m	Active constra-int/s
	bf_1	tf_1	bf_2	tf_2	hw	tw		
1.0	50.0	6.4	50.0	6.4	100.0	4.0	8.16	$g_1^{\oplus}$, $g_3^{\oplus\oplus}$
1.5	68.0	6.5	68.0	6.5	118.0	4.3	10.88	none
2.0	82.4	6.8	82.4	6.8	156.0	4.9	14.80	g_{15}^{**}

$\oplus$Lower bound on the top flange width.

$\oplus\oplus$Lower bound on the bottom flange width.

**Maximum deflection in the beam.

Table 6.12

Optimal Sectional Dimensions

Loading: linearly varying

Intensity: 0–3000 kg/m ; Lateral Eccentricity: nil

Span in m	Fixed Supports								Simple Supports							
	Optimal dimensions in mm						Optimum weight in kg/m	Active constraint/s	Optimal dimensions in mm						Optimum weight in kg/m	Active constraint/s
	bf_1	tf_1	bf_2	tf_2	hw	tw			bf_1	tf_1	bf_2	tf_2	hw	tw		
3.0	76.8	6.6	76.8	6.6	134.0	4.5	12.76	g_{13}^{*}	82.4	6.8	82.4	6.8	156.0	4.9	14.80	g_{15}^{**}
4.0	92.4	7.0	92.4	7.0	181.0	5.2	17.49	g_{13}	100.0	7.6	100.0	7.6	205.0	5.5	20.69	g_{15}
5.0	106.0	8.6	106.0	8.6	231.0	5.9	25.01	g_{13}	120.0	8.7	120.0	8.7	245.0	6.0	27.97	g_{15}
6.0	132.2	8.8	132.2	8.8	262.0	6.2	30.99	g_{13}	144.8	9.1	144.8	9.1	287.0	6.5	35.38	g_{15}

*Maximum normal stress in the beam.

**Maximum deflection in the beam.

Table 6.13

Optimal Sectional Dimensions

Loading: linearly varying

Intensity: 3000-0 kg/m ; Lateral Eccentricity: nil

Span in m	Cantilever							
	Optimal dimensions in mm						Optimum weight in kg/m	Active constra-int/s
	bf_1	tf_1	bf_2	tf_2	hw	tw		
1.0	50.0	6.4	50.0	6.4	100.0	4.0	8.16	$g_1^{\oplus}$, $g_3^{\oplus\oplus}$
1.5	76.6	6.6	76.6	6.6	133.0	4.5	12.67	g_{15}^{**}
2.0	91.6	7.0	91.6	7.0	179.0	5.2	17.26	g_{15}

$\oplus$Lower bound on the top flange width.

$\oplus\oplus$Lower bound on the bottom flange width.

**Maximum deflection in the beam.

Table 6.14

Optimal Sectional Dimensions

Loading: linearly varying

Intensity: 1000-2000 kg/m

Lateral Eccentricity: nil

Span in m	Fixed Supports: Optimal dimensions in mm						Fixed Supports: Optimum weight in kg/m	Fixed Supports: Active constraint/s	Simple Supports: Optimal dimensions in mm						Simple Supports: Optimum weight in kg/m	Simple Supports: Active constraint/s
	bf_1	tf_1	bf_2	tf_2	hw	tw			bf_1	tf_1	bf_2	tf_2	hw	tw		
3.0	75.0	6.5	75.0	6.5	125.0	4.4	11.97	g_{13}^{*}	82.4	6.8	82.4	6.8	156.0	4.9	14.80	g_{15}^{**}
4.0	88.8	6.9	88.8	6.9	172.0	5.1	16.44	g_{13}	100.0	7.6	100.0	7.6	205.0	5.5	20.69	g_{15}
5.0	101.0	8.6	101.0	8.6	226.0	5.8	23.96	g_{13}	120.0	8.7	120.0	8.7	245.0	6.0	27.97	g_{15}
6.0	126.8	8.7	126.8	8.7	253.0	6.1	29.55	g_{13}	144.8	9.1	144.8	9.1	287.0	6.5	35.38	g_{15}

*Maximum normal stress in the beam.

**Maximum deflection in the beam.

Table 6.15

Optimal Sectional Dimensions

Loading: linearly varying

Intensity: 2000-1000 kg/m ; Lateral Eccentricity: nil

Span in m	Cantilever							
	Optimal dimensions in mm						Optimum weight in kg/m	Active constra-int/s
	bf_1	tf_1	bf_2	tf_2	hw	tw		
1.0	57.0	6.4	57.0	6.4	107.0	4.1	9.20	g_{15}^{**}
1.5	82.8	6.8	82.8	6.8	157.0	4.9	14.89	g_{15}
2.0	100.0	7.6	100.0	7.6	206.0	5.5	20.83	g_{15}

**Maximum deflection in the beam.

Table 6.18

Optimal Sectional Dimensions

Loading: linearly varying

Intensity: 2000-3000 kg/m ; Lateral Eccentricity: nil

Span in m	Fixed Supports								Simple Supports							
	Optimal dimensions in mm						Optimum weight in kg/m	Active constraint/s	Optimal dimensions in mm						Optimum weight in kg/m	Active constraint/s
	bf_1	tf_1	bf_2	tf_2	hw	tw			bf_1	tf_1	bf_2	tf_2	hw	tw		
3.0	85.2	6.9	85.2	6.9	163.0	5.0	15.52	g_{13}^{*}	97.6	7.2	97.6	7.2	194.0	5.3	19.17	g_{13}
4.0	100.0	7.9	100.0	7.9	212.0	5.6	21.74	g_{13}	116.0	8.7	116.0	8.7	241.0	6.0	27.12	g_{13}
5.0	123.0	8.7	123.0	8.7	248.0	6.1	28.64	g_{13}	145.2	9.1	145.2	9.1	288.0	6.6	35.62	g_{13}
6.0	146.0	9.2	146.0	9.2	290.0	6.6	36.07	g_{13}	165.0	10.1	165.0	10.1	330.0	7.1	44.56	g_{13}, g_{15}^{**}

*Maximum normal stress in the beam.

**Maximum deflection in the beam.

Table 6.19

Optimal Sectional Dimensions

Loading: linearly varying

Intensity: 3000-2000 kg/m ; Lateral Eccentricity: nil

Span in m	Cantilever							
	Optimal dimensions in mm						Optimum weight in kg/m	Active constra-int/s
	bf_1	tf_1	bf_2	tf_2	hw	tw		
1.0	75.0	6.5	75.0	6.5	125.0	4.4	11.97	g_{15}^{**}
1.5	95.6	7.1	95.6	7.1	189.0	5.3	18.52	g_{13}^{*}, g_{15}
2.0	116.0	8.7	116.0	8.7	241.0	6.0	27.12	g_{15}

*Maximum normal stress in the beam.

**Maximum deflection in the beam.

Table 6.20

Optimal Sectional Dimensions

Loading: nodal loads

Intensity: 1000 kg/m (equivalent) Lateral Eccentricity: nil

Span in m	Fixed Supports								Simple Supports							
	Optimal dimensions in mm						Optimum weight in kg/m	Active constraint/s	Optimal dimensions in mm						Optimum weight in kg/m	Active constraint/s
	bf_1	tf_1	bf_2	tf_2	hw	tw			bf_1	tf_1	bf_2	tf_2	hw	tw		
3.0	50.0	6.4	50.0	6.4	100.0	4.0	5.16	$g_1^{\oplus}, g_3^{\oplus\oplus}$	75.0	6.5	75.0	6.5	125.0	4.4	11.97	none
4.0	70.0	6.5	70.0	6.5	120.0	4.3	11.19	g_{13}^{*}	87.6	6.9	37.6	6.9	169.0	5.0	16.14	g_{15}^{**}
5.0	88.0	6.9	8[illegible].0	6.9	170.0	5.0	16.23	g_{13}	100.0	8.3	100.0	8.3	220.0	5.7	22.97	g_{13}
6.0	100.0	7.9	100.0	7.9	212.0	5.6	21.74	g_{13}	124.0	8.7	124.0	8.7	249.0	6.1	28.84	g_{13}

⊕Lower bound on the top flange width.

⊕⊕Lower bound on the bottom flange width.

*Maximum normal stress in the beam.

**Maximum deflection in the beam.

Table 6.21

Optimal Sectional Dimensions

Loading: nodal loads

Intensity: 2000 kg/m (equivalent); Lateral Eccentricity: nil

Span in m	Fixed Supports								Simple Supports							
	Optimal dimensions in mm						Optimum weight in kg/m	Active constraint/s	Optimal dimensions in mm						Optimum weight in kg/m	Active constraint/
	bf_1	tf_1	bf_2	tf_2	hw	tw			bf_1	tf_1	bf_2	tf_2	hw	tw		
3.0	72.0	6.5	72.0	6.5	122.0	4.3	11.50	g_{13}*	90.4	6.9	90.4	6.9	176.0	5.1	16.87	g_{13}
4.0	86.8	6.9	86.8	6.9	167.0	5.0	15.92	g_{13}	100.0	8.6	100.0	8.6	225.0	5.8	23.75	g_{13}
5.0	100.0	8.3	100.0	8.3	220.0	5.7	22.97	g_{13}	132.8	8.8	132.8	8.8	263.0	6.3	31.19	g_{13}
6.0	124.0	8.7	124.0	8.7	249.0	6.1	28.84	g_{13}	152.4	9.5	152.4	9.5	304.0	6.7	38.77	g_{13}

*Maximum normal stress in the beam.

Table 6.22

Optimal Sectional Dimensions

Loading: nodal loads

Intensity: 3000 kg/m (equivalent); Lateral Eccentricity: nil

Span in m	Fixed Supports								Simple Supports							
	Optimal dimensions in mm						Optimum weight in kg/m	Active constraint/s	Optimal dimensions in mm						Optimum weight in kg/m	Active constraint/s
	bf_1	tf_1	bf_2	tf_2	hw	tw			bf_1	tf_1	bf_2	tf_2	hw	tw		
3.0	80.0	6.8	80.8	6.8	152.0	4.8	14.39	g_{13}*	100.0	7.8	100.0	7.8	209.0	5.5	21.29	g_{13}
4.0	99.6	7.3	99.6	7.3	199.0	5.4	19.82	g_{13}	129.8	8.7	129.8	8.7	258.0	6.2	30.37	g_{13}
5.0	115.0	8.7	115.0	8.7	240.0	6.0	26.90	g_{13}	155.4	9.5	155.4	9.5	309.0	6.8	39.82	g_{13}
6.0	141.6	8.9	141.6	8.9	279.0	6.5	33.87	g_{13}	165.0	11.4	165.0	11.4	351.0	7.4	50.00	g_{13}

*Maximum normal stress in the beam.

Table 6.23

Optimal Sectional Dimensions

Loading: uniformly distributed

Intensity: 1000 kg/m ; Lateral Eccentricity: 0.1 bf_1 (from Table 6.2)

Span in m	Fixed Supports								Simple Supports							
	Optimal dimensions in mm						Optimum weight in kg/m	Active constraint/s	Optimal dimensions in mm						Optimum weight in kg/m	Active constraint/s
	bf_1	tf_1	bf_2	tf_2	hw	tw			bf_1	tf_1	bf_2	tf_2	hw	tw		
3.0	69.0	6.5	69.0	6.5	119.0	4.3	11.05	g_{13}*	77.0	6.6	77.0	6.6	135.0	4.6	12.84	g_{15}**
4.0	86.0	6.9	86.0	6.9	165.0	5.0	15.71	g_{13}	92.4	7.0	92.4	7.0	181.0	5.2	17.49	g_{15}
5.0	100.0	8.4	100.0	8.4	221.0	5.7	23.11	g_{13}	104.0	8.6	104.0	8.6	229.0	5.9	24.59	g_{13}
6.0	124.0	8.7	124.0	8.7	249.0	6.1	28.84	g_{13}	132.8	8.8	132.8	8.8	263.0	6.3	31.19	g_{13}

*Maximum normal stress in the beam.

**Maximum deflection in the beam.

Table 6.24

Optimal Sectional Dimensions

Loading: uniformly distributed

Intensity: 1000 kg/m ; Lateral Eccentricity: 0.2 bf_1 (from Table 6.2)

Span in m	Fixed Supports								Simple Supports							
	Optimal dimensions in mm						Optimum weight in kg/m	Active constraint/s	Optimal dimensions in mm						Optimum weight in kg/m	Active constraint/s
	bf_1	tf_1	bf_2	tf_2	hw	tw			bf_1	tf_1	bf_2	tf_2	hw	tw		
3.0	76.4	6.6	76.4	6.6	132.0	4.5	12.57	g_{13}*	78.6	6.7	78.6	6.7	143.0	4.7	13.56	g_{13}
4.0	94.8	7.1	94.8	7.1	187.0	5.2	18.24	g_{13}	98.0	7.2	98.0	7.2	195.0	5.3	19.28	g_{13}
5.0	110.0	8.6	110.0	8.6	235.0	5.9	25.84	g_{13}	111.0	8.7	111.0	8.7	236.0	5.9	26.06	g_{13}
6.0	135.2	8.8	135.2	8.8	267.0	6.3	31.82	g_{13}	141.2	8.9	141.2	8.9	278.0	6.4	33.70	g_{13}

*Maximum normal stress in the beam.

Table 6.25

Optimal Sectional Dimensions

Loading: uniformly distributed

Intensity: 1000 kg/m ;

Lateral Eccentricity: 0.3 bf_1 (from Table 6.2)

Span in m	Fixed Supports								Simple Supports							
	Optimal dimensions in mm						Optimum weight in kg/m	Active constraint/s	Optimal dimensions in mm						Optimum weight in kg/m	Active constraint/s
	bf_1	tf_1	bf_2	tf_2	hw	tw			bf_1	tf_1	bf_2	tf_2	hw	tw		
3.0	79.8	6.8	79.8	6.8	149.0	4.8	14.11	g_{13}^*	82.4	6.8	82.4	6.8	156.0	4.9	14.80	g_{13}
4.0	100.0	7.5	100.0	7.5	204.0	5.5	20.53	g_{13}	100.0	7.7	100.0	7.7	207.7	5.5	20.98	g_{13}
5.0	119.0	8.7	119.0	8.7	244.0	6.0	27.70	g_{13}	117.0	8.7	117.0	8.7	242.0	6.0	27.32	g_{13}
6.0	143.6	9.0	143.6	9.0	284.0	6.5	34.76	g_{13}	146.0	9.2	146.0	9.2	290.0	6.6	36.07	g_{13}

*Maximum normal stress in the beam.

Table 6.28

Optimal Sectional Dimensions

Loading: uniformly distributed

Intensity: 2000 kg/m ;

Lateral Eccentricity: 0.1 bf_1 (from Table 6.4)

Span in m	Fixed Supports								Simple Supports							
	Optimal dimensions in mm						Optimum weight in kg/m	Active constraint/s	Optimal dimensions in mm						Optimum weight in kg/m	Active constraint/s
	bf_1	tf_1	bf_2	tf_2	hw	tw			bf_1	tf_1	bf_2	tf_2	hw	tw		
3.0	91.2	7.0	91.2	7.0	178.0	5.1	17.13	g_{13}^*	99.2	7.3	99.2	7.3	198.0	5.4	19.68	g_{13}
4.0	102.0	8.6	102.0	8.6	227.0	5.8	24.16	g_{13}	117.0	8.7	117.0	8.7	242.0	6.0	27.32	g_{13}
5.0	133.4	8.8	133.4	8.8	264.0	6.3	31.34	g_{13}	147.6	9.3	147.6	9.3	294.0	6.6	36.81	g_{13}
6.0	154.8	9.5	154.8	9.5	308.0	6.8	39.60	g_{13}	165.0	10.5	165.0	10.5	335.0	7.2	46.13	g_{13}

*Maximum normal stress in the beam.

Table 6.29

Optimal Sectional Dimensions

Loading: uniformly distributed

Intensity: 2000 kg/m ;

Lateral Eccentricity: 0.2 bf_1 (from Table 6.4)

Span in m	Fixed Supports								Simple Supports							
	Optimal dimensions in mm						Optimum weight in kg/m	Active constraint/s	Optimal dimensions in mm						Optimum weight in kg/m	Active constraint/s
	bf_1	tf_1	bf_2	tf_2	hw	tw			bf_1	tf_1	bf_2	tf_2	hw	tw		
3.0	100.0	7.3	100.0	7.3	200.0	5.4	19.94	g_{13}*	100.0	8.1	100.0	8.1	216.0	5.6	22.34	g_{13}
4.0	122.0	3.7	122.0	8.7	247.0	6.1	28.39	g_{13}	132.8	8.8	132.8	8.8	263.0	6.3	31.19	g_{13}
5.0	147.2	9.3	147.2	9.3	293.0	6.6	36.63	g_{13}	163.2	9.8	163.2	9.8	322.0	7.0	42.57	g_{13}
6.0	165.0	10.8	165.0	10.8	340.0	7.2	47.19	g_{13}	165.0	11.7	165.0	11.7	366.0	7.6	52.25	g_{13}

*Maximum normal stress in the beam.

Table 6.30

Optimal Sectional Dimensions

Loading: uniformly distributed

Intensity: 2000 kg/m ;

Lateral Eccentricity: 0.3 bf_1 (from Table 6.4)

Span in m	Fixed Supports								Simple Supports							
	Optimal dimensions in mm						Optimum weight in kg/m	Active constraint/s	Optimal dimensions in mm						Optimum weight in kg/m	Active constraint/s
	bf_1	tf_1	bf_2	tf_2	hw	tw			bf_1	tf_1	bf_2	tf_2	hw	tw		
3.0	100.0	8.3	100.0	8.3	220.0	5.7	22.97	g_{13}*	106.0	8.6	106.0	8.6	231.0	5.9	25.01	g_{13}
4.0	134.6	8.8	134.6	8.8	266.0	6.3	31.67	g_{13}	144.0	9.0	144.0	9.0	285.0	6.5	34.93	g_{13}
5.0	159.0	9.6	159.0	9.6	315.0	6.9	41.08	g_{13}	165.0	10.0	165.0	10.9	341.0	7.2	47.56	g_{13}
6.0	165.0	12.1	165.0	12.1	380.0	7.8	54.39	g_{13}	165.8	12.7	165.8	12.7	408.0	8.1	58.87	g_{13}

*Maximum normal stress in the beam.

Table 6.31

Optimal Sectional Dimensions

Loading: uniformly distributed

Intensity: 2000 kg/m ; Lateral Eccentricity: 0.1 bf_1

(from Table 6.5)

Span in m	Cantilever							
	Optimal dimensions in mm						Optimum weight in kg/m	Active constraint/s
	bf_1	tf_1	bf_2	tf_2	hw	tw		
1.0	77.6	6.7	77.6	6.7	138.0	4.6	13.11	g_{13}^{*}
1.5	100.0	7.7	100.0	7.7	208.0	5.5	21.13	g_{13}
2.0	126.8	8.7	126.8	8.7	253.0	6.1	29.55	g_{13}

*Maximum normal stress in the beam.

Table 6.32

Optimal Sectional Dimensions

Loading: uniformly distributed

Intensity: 2000 kg/m ; Lateral Eccentricity: 0.2 bf_1

(from Table 6.5)

Span in m	Cantilever							
	Optimal dimensions in mm						Optimum weight in kg/m	Active constra-int/s
	bf_1	tf_1	bf_2	tf_2	hw	tw		
1.0	83.6	6.8	83.6	6.8	159.0	4.9	15.11	g_{13}^*
1.5	105.0	8.6	105.0	8.6	230.0	5.9	24.79	g_{13}
2.0	142.4	8.9	142.4	8.9	281.0	6.5	34.21	g_{13}

*Maximum normal stress in the beam.

Table 6.33

Optimal Sectional Dimensions

Loading: uniformly distributed

Intensity: 3000 kg/m ;

Lateral Eccentricity: 0.1 bf_1 (from Table 6.6)

Span in m	Fixed Supports								Simple Supports							
	Optimal dimensions in mm						Optimum weight in kg/m	Active constraint/s	Optimal dimensions in mm						Optimum weight in kg/m	Active constraint/s
	bf_1	tf_1	bf_2	tf_2	hw	tw			bf_1	tf_1	bf_2	tf_2	hw	tw		
3.0	100.0	8.0	100.0	8.0	214.0	5.6	22.05	g_{13}^*	109.0	8.6	109.0	8.6	234.0	5.9	25.64	g_{13}
4.0	129.8	8.7	129.8	8.7	258.0	6.2	30.37	g_{13}	148.0	9.3	148.0	9.3	295.0	6.6	36.99	g_{13}
5.0	158.4	9.6	158.4	9.6	314.0	6.9	40.88	g_{13}	165.0	11.1	165.0	11.1	345.0	7.3	48.52	g_{13}
6.0	165.0	11.6	165.0	11.6	361.0	7.5	51.49	g_{13}	166.1	12.7	166.1	12.7	411.0	8.1	59.35	g_{13}

*Maximum normal stress in the beam.

Table 6.34

Optimal Sectional Dimensions

Loading: uniformly distributed

Intensity: 3000 kg/m ;

Lateral Eccentricity: 0.2 bf_1 (from Table 6.6)

Span in m	Fixed Supports								Simple Supports							
	Optimal dimensions in mm						Optimum weight in kg/m	Active constraint/s	Optimal dimensions in mm						Optimum weight in kg/m	Active constraint/s
	bf_1	tf_1	bf_2	tf_2	hw	tw			bf_1	tf_1	bf_2	tf_2	hw	tw		
3.0	111.0	8.7	111.0	8.7	236.0	5.9	26.06	g_{13}*	128.0	8.7	128.0	8.7	255.0	6.2	29.85	g_{13}
4.0	144.8	9.1	144.8	9.1	287.0	6.5	35.38	g_{13}	164.4	9.8	164.4	9.8	324.0	7.0	43.05	g_{13}
5.0	165.0	11.1	165.0	11.1	345.0	7.3	48.52	g_{13}	165.0	12.3	165.0	12.3	393.0	7.9	56.40	g_{13}
6.0	168.0	13.0	168.0	13.0	430.0	8.4	62.61	g_{13}	171.4	14.0	171.4	14.0	457.0	8.7	68.67	g_{13}

*Maximum normal stress in the beam.

Table 6.35

Optimal Sectional Dimensions

Loading: uniformly distributed

Intensity: 3000 kg/m ; Lateral Eccentricity: 0.3 bf_1 (from Table 6.6)

Span in m	Fixed Supports								Simple Supports							
	Optimal dimensions in mm						Optimum weight in kg/m	Active constraint/s	Optimal dimensions in mm						Optimum weight in kg/m	Active constraint/s
	bf_1	tf_1	bf_2	tf_2	hw	tw			bf_1	tf_1	bf_2	tf_2	hw	tw		
3.0	125.0	8.7	125.0	8.7	250.0	6.1	29.04	g_{13}*	140.8	8.8	140.8	8.8	277.0	6.4	33.50	g_{13}
4.0	154.8	9.5	154.8	9.5	308.0	6.8	39.60	g_{13}	165.0	11.1	165.0	11.1	345.0	7.3	48.52	g_{13}
5.0	165.0	12.3	165.0	12.3	390.0	7.9	55.94	g_{13}	168.9	13.2	168.9	13.2	439.0	8.5	64.22	g_{13}
6.0	174.0	14.4	174.0	14.4	470.0	8.8	71.67	g_{13}	183.2	14.4	183.2	14.4	513.0	9.4	79.29	g_{13}

*Maximum normal stress in the beam.

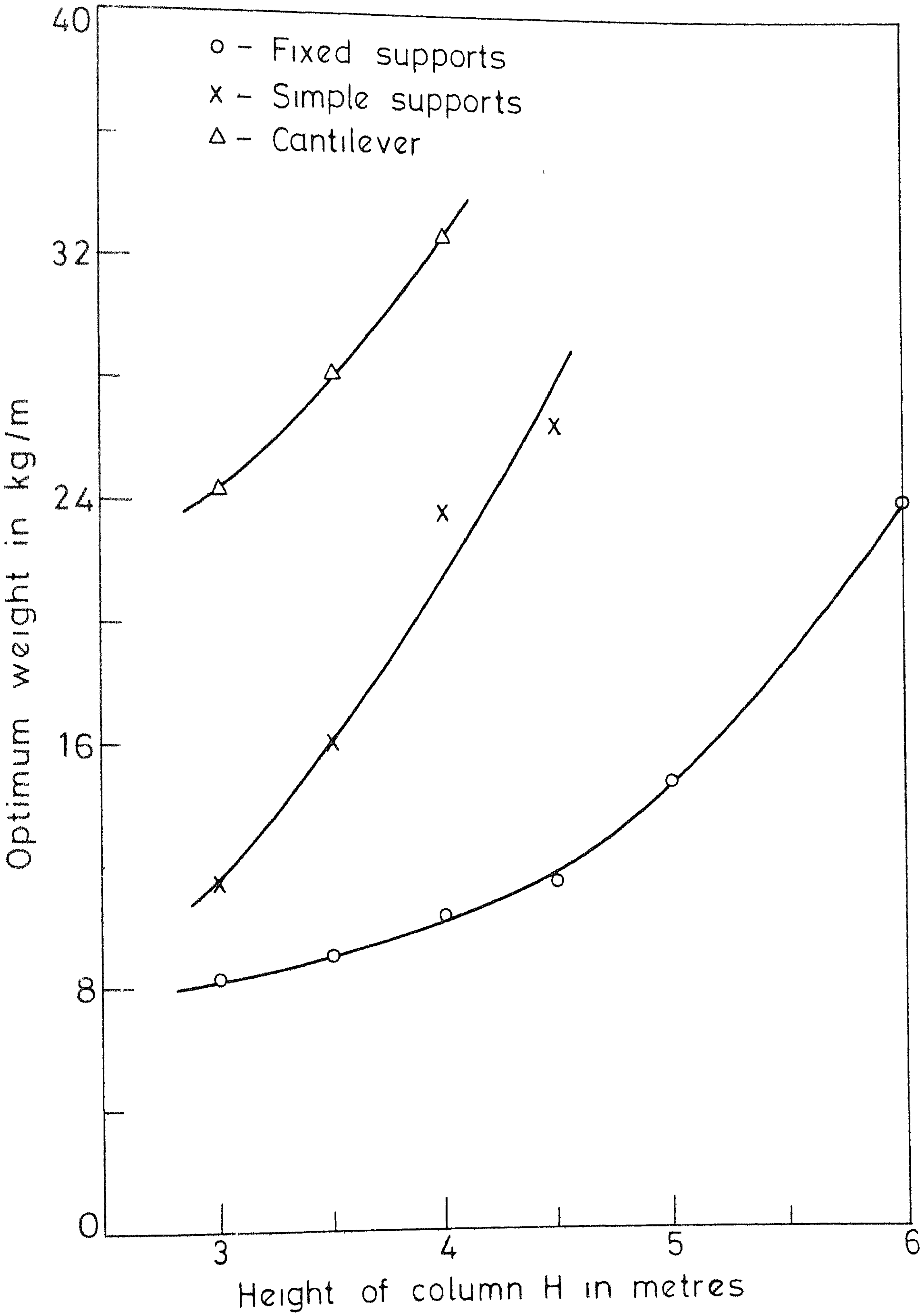

Fig. 6.1 Optimum weight of column sections for various heights and boundary conditions.

Table 6.37

Optimal Sectional Dimensions

Loading: uniformly distributed

Intensity: 3000 kg/m ; Lateral Eccentricity: 0.2 bf_1

(from Table 6.7)

Span in m	Cantilever							
	Optimal dimensions in mm						Optimum weight in kg/m	Active constra-int/s
	bf_1	tf_1	bf_2	tf_2	hw	tw		
1.5	98.4	7.2	98.4	7.2	196.0	5.3	19.40	g_{13}^{*}
1.5	133.4	8.8	133.4	8.8	264.0	6.3	31.34	g_{13}
2.0	165.0	10.3	165.0	10.3	333.0	7.2	45.40	g_{13}

*Maximum normal stress in the beam.

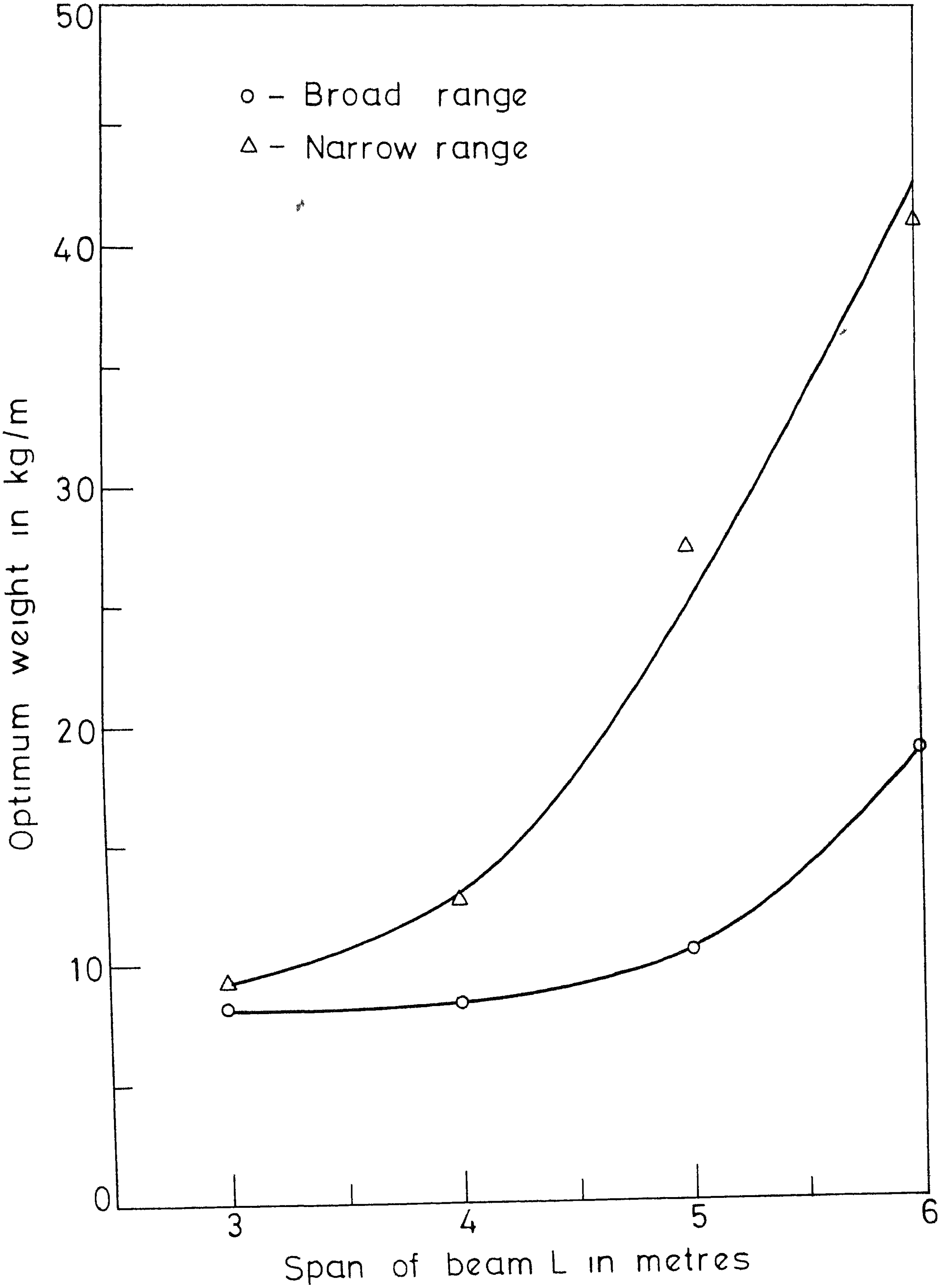

Fig. 6.3 Optimum weight of simple beams for various spans and frequency ranges.

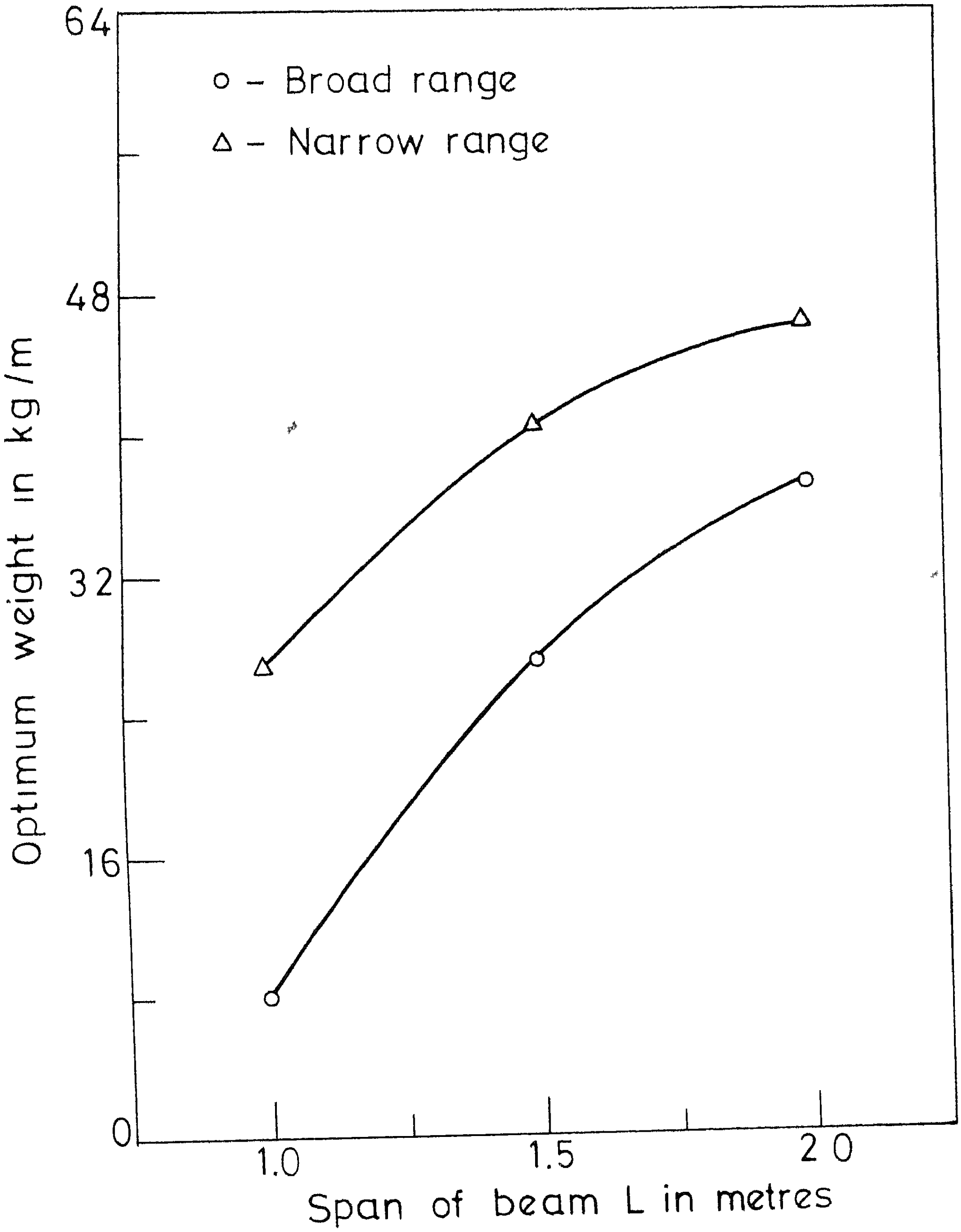

Fig. 6.4 Optimum weight of cantilever beams for various spans and frequency ranges.

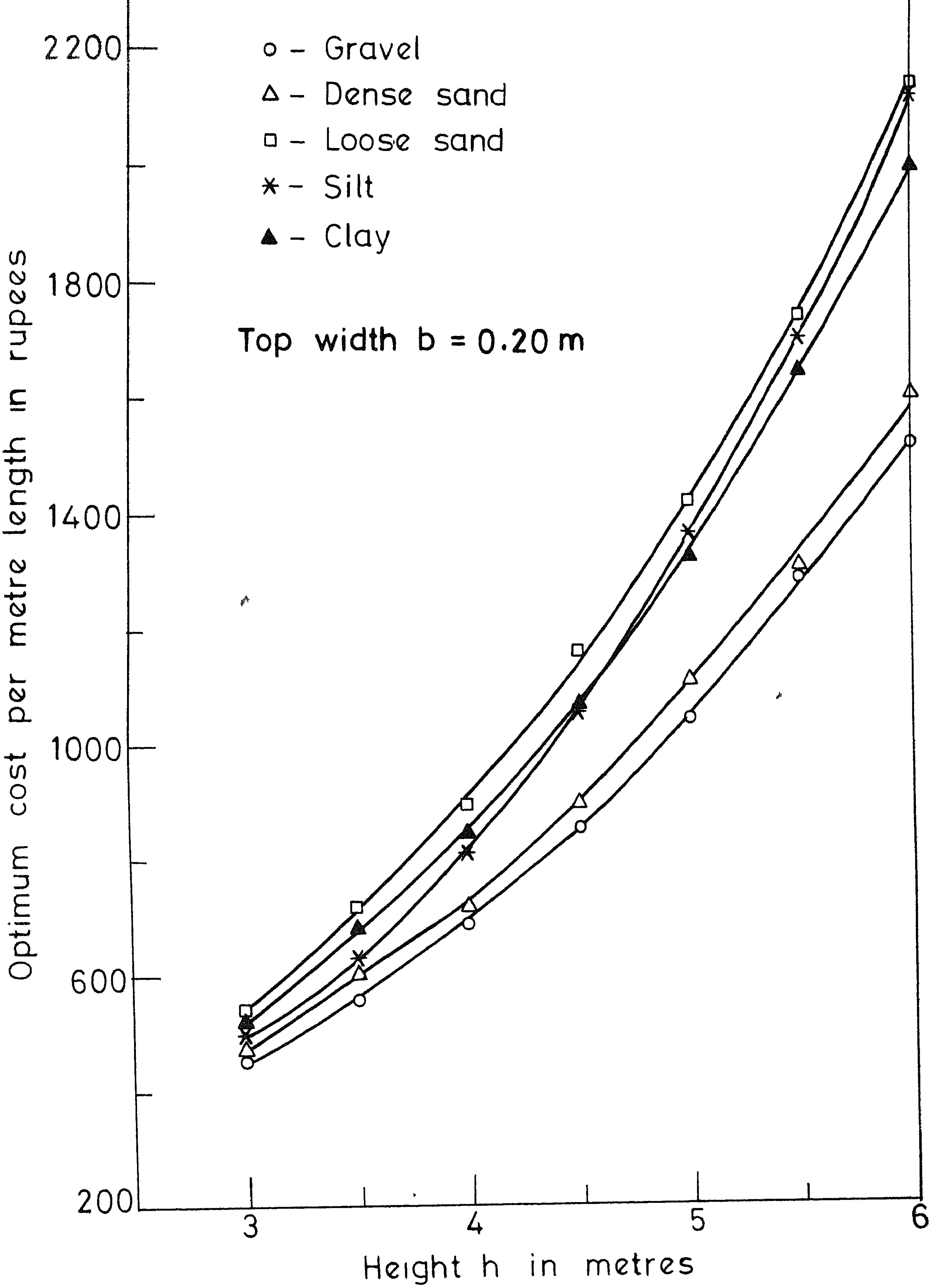

Fig. 6.5 Optimum cost of reinforced concrete cantilever retaining walls for various heights and soils.

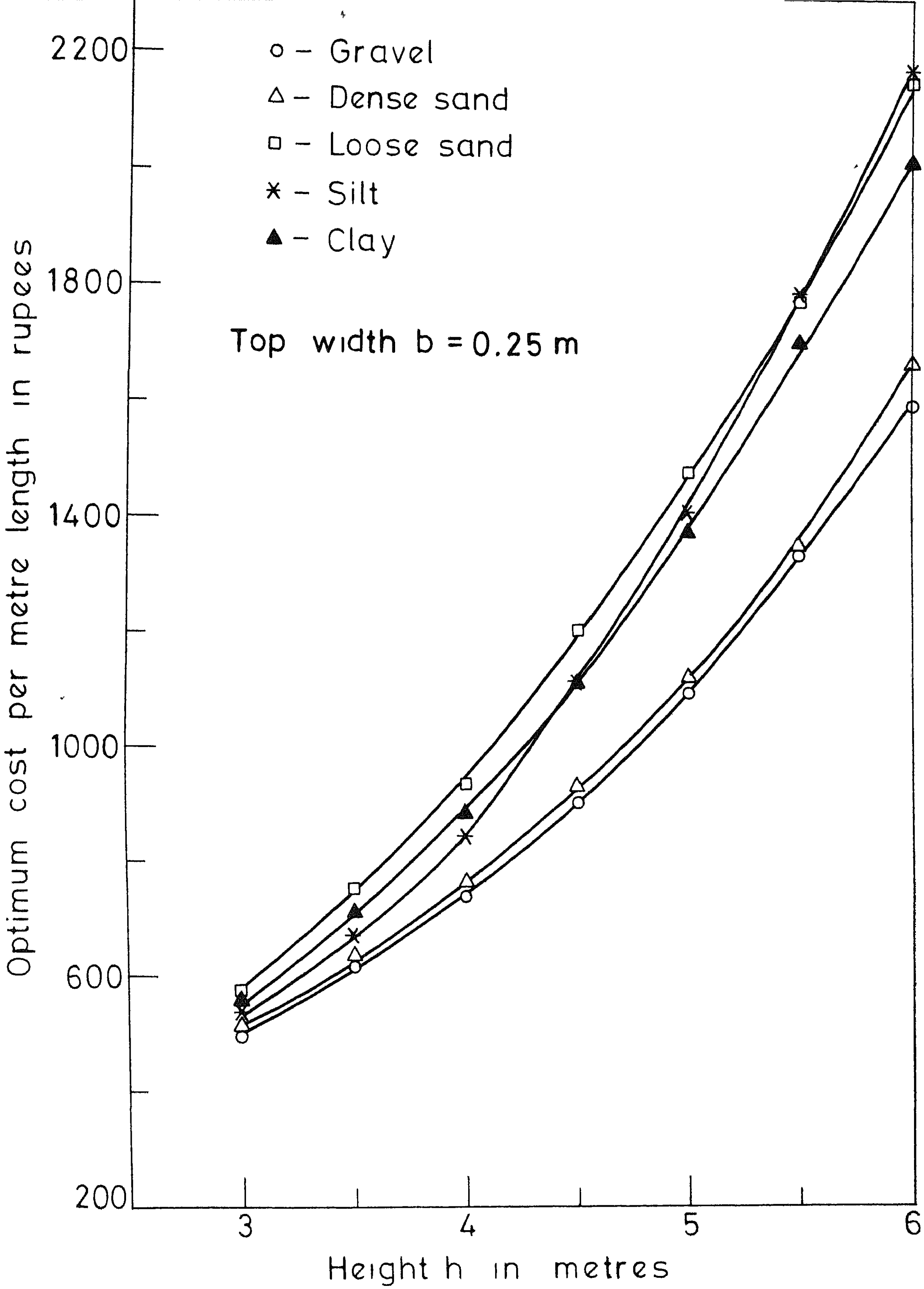

Fig. 6.6 Optimum cost of reinforced concrete cantilever retaining walls for various heights and soils.

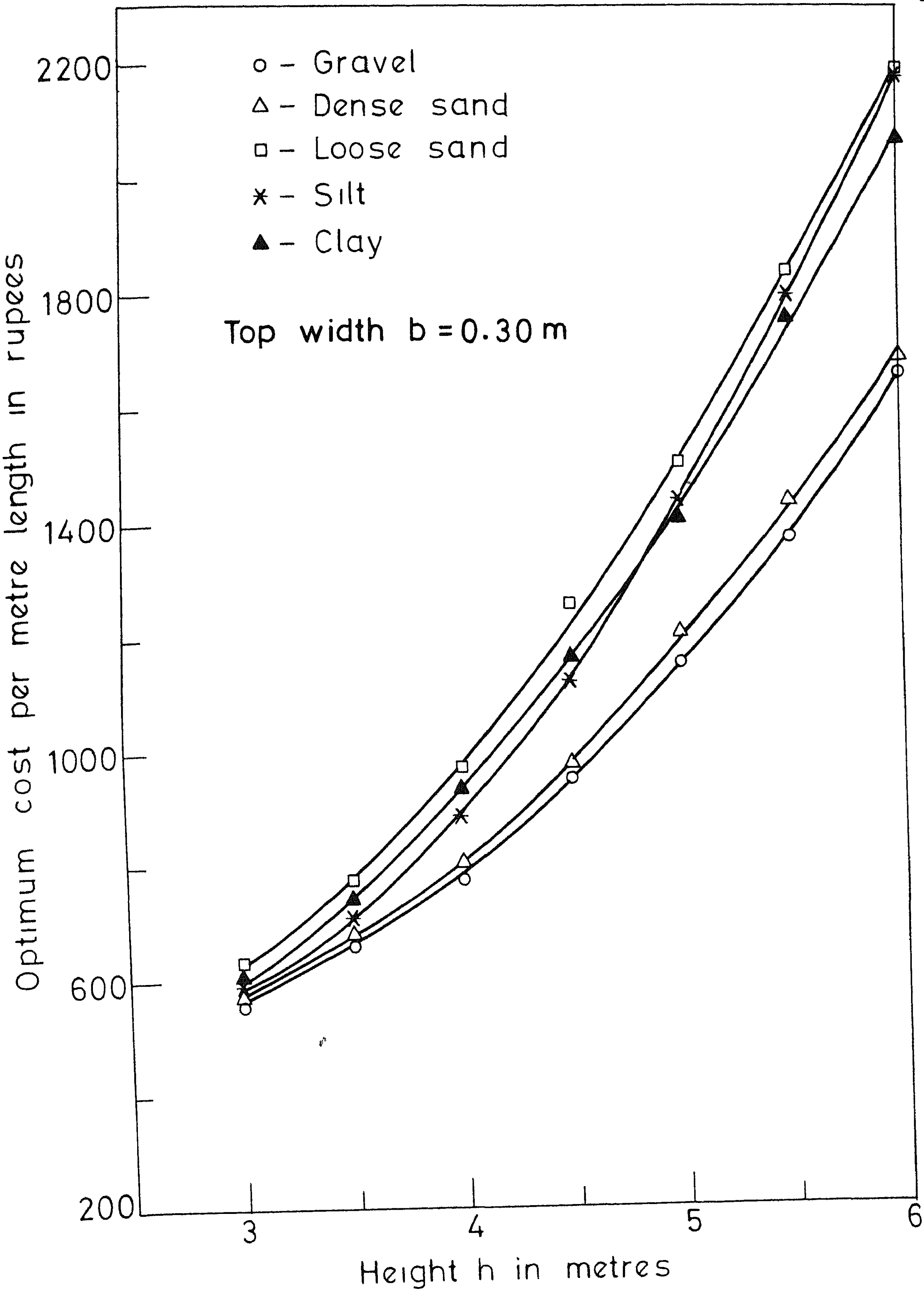

Fig. 6.7 Optimum cost of reinforced concrete cantilever retaining walls for various heights and soils.

LIST OF REFERENCES

Adidam, S.R. 1972 Plastic analysis and optimal design of plates and shells, Ph.D. Thesis, Monash University, Clayton, Victoria, Australia

Archer, J.S. 1963 Consistent matrix analysis for distributed mass systems, J. Struct. Div., Proc. ASCE, Vol. 89, No. ST4, pp. 161-78

Archer, J.S. 1965 Consistent matrix formulations for structural analysis using finite element techniques, AIAA J., Vol. 3, No. 10, pp. 1910-18

Arora, J.S. and Haug, E.J. 1976 Efficient optimal design of structures by generalized steepest descent programming, Int. J. Num. Meth. Eng., Vol. 10, No. 4, pp. 747-66

Barnett, R.L. 1961 Minimum weight design of beams for deflections, J. Eng. Mech. Div., Proc. ASCE, Vol. 87, No. EM1, pp. 75-109

Barsoum, R.S. and Gallagher, R.H. 1970 Finite element analysis of torsional-flexural stability problems, Int. J. Num. Meth. Eng., Vol. 2, No. 3, pp. 335-52

Bleich, F. 1952 Buckling strength of metal structures, McGraw-Hill, New York

Box, M.J., Davies, D. and Swann, W.H. 1969 Non-linear optimization techniques, ICI Ltd., Monograph No. 5, Oliver and Boyd, Edinburgh

Bronowicki, A.J. and Felton, L.P. 1975 Optimum design of continuous thin walled beams, Int. J. Num. Meth. Eng., Vol. 9, No. 3, pp. 711-20

Carnegie, W., Thomas, J. and Dokumaci, E. 1969 An improved method of matrix displacement analysis in vibration problems, Aero. Quart., Vol. XX, Part 4, pp. 321-32

Gallagher, R.H. and Padlog, R.J.	1963	Discrete element approach to structural instability analysis, AIAA J., Vol. 1, No. 6, pp. 1437-39
Gallagher, R.H., Gellatly, R.A., Padlog, R.J. and Mallett, R.H.	1967	A discrete element procedure for thin shell instability analysis, AIAA J., Vol. 5, No. 1, pp. 138-45
Gallagher, R.H. and Zienkiewicz, O.C., Ed. by	1973	Optimum structural design: theory and applications, John Wiley, New York
Gallagher, R.H.	1975	Finite element analysis: fundamentals, Prentice-Hall, Englewood Cliffs, New Jersey
Gere, J.M. and Weaver, W.	1969	Analysis of framed structures, Van Nostrand, Princeton, New Jersey
Goldberg, J.E., Bogdenoiff, J.E. and Glanz, W.D.	1964	Lateral and torsional buckling of thin walled beams, Publ. Int. Assn. Bridge Struct. Eng., Vol. 24, pp. 92
Gourlay, A.R. and Watson, G.A.	1973	Computational methods for matrix eigenvalues, John Wiley, New York
Gupta, M. and Friel, L.	1977	Design aids for cantilever retaining walls, J. Struct. Div., Proc. ASCE, Vol. 103, No. ST5, pp. 1113-26
Haaijer, G.	1961	Economy of high strength steel structural members, J. Struct. Div., Proc. ASCE, Vol. 87, No. ST8, pp. 1-23
Haftka, R.T. and Starnes, J.H.	1976	Application of a quadratic extended interior penalty function to structural optimization, AIAA J., Vol. 14, No. 6, pp. 718-24
Hairshine, R.C.	1972	A design chart for determining the optimum base proportions of free standing retaining walls, Proc. Inst Civil Eng., Vol. 51, pp. 295-318
Hartz, Billy J.	1965	Matrix formulation of structural stability problems, J. Struct. Div., Proc. ASCE, Vol. 91, No. ST6, pp. 141-57

Hill, S.	1971	The design of cantilever retaining walls in cohesive soils, Civil Eng. Pub. Works Rev., Vol. 66, No. 784, pp. 1210-13
Holden, J.T.	1972	On the finite deflection of thin beams, Int. J. Solids Struct., Vol. 8, No. 8, pp. 1051-55
Hooke, R. and Jeeves, T.A.	1961	Direct search solution of numerical and statistical problems, J. Assoc. Comp. Mach., Vol. 8, No. 2, pp. 212-29
Horne, M.R., Hoh, K.H. and Poskitt, T.J.	1971	The torsional flexural behaviour of thin walled prismatic members, Int. J. Mech. Sci., Vol. 13, No. 7, pp. 641-57
Huntington, W.C.	1967	Earth pressure and retaining walls, John Wiley, New York
IS : 456	1964	Code of practice for plain and reinforced concrete
IS : 800	1962	Code of practice for use of structural steel in general building construction
IS : 1498	1970	Classification and identification of soils for general engineering purposes
Johnston, B.G.	1966	Guide to design criteria for material compression members, John Wiley, New York
Kamat, M.P. and Simitses, G.J.	1973	Optimal beam frequencies by the finite element displacement method, Int. J. Solids Struct., Vol. 9, No. 3, pp. 415-29
Kavlie, D. and Moe, J.	1971	Automated design of frame structures, J. Struct. Div., Proc. ASCE, Vol. 97, No. ST1, pp. 33-62
Kawai, T., Muraki, T., Tanaka, N. and Iwaki, T.	1971	Finite element analysis of thin walled structures based on modern engineering theory of beams, Proc. 3rd Conf. Matrix Methods Struct. Mech., Wright-Patterson Air Force Base, Ohio, pp. 113-48

Kelly, J.E.	1960	The cutting plane method for solving convex problems, SIAM J., Vol. VIII, No. 4, pp. 703-12
Kennedy, J.B. and Murty, M.K.S.	1972	Buckling of steel angle and tee struts, J. Struct. Div., Proc. ASCE, Vol. 98, No. ST11, pp. 2507-22
Khot, N.S., Venkayya, V.B. and Berke, L.	1976	Optimum structural design with stability constraints, Int. J. Num. Meth. Eng., Vol. 10, No. 5, pp.1097-1114
Kollbrunner, F. and Hajdiu, N.	1968	Displacement method in the theory of thin walled members and a new calculation model for the thin walled bars with deformed contours, Publ. Int. Assn. Bridge Struct. Eng., Vol. 28-II, pp. 87-100
Krahula, J.L.	1967	Analysis of bent and twisted bars using the finite element method, AIAA J., Vol. 5, No. 6, pp. 1194-97
Krajcinovic, D.	1969	A consistent discrete element technique for thin walled assemblages, Int. J. Solids Struct., Vol. 5, No. 9, pp. 639-62
Krajcinovic, D.	1970	Matrix force analysis of thin walled structures, J. Struct. Div., Vol. 96, No. ST1, pp. 107-21
Lay, M.G.	1965	Flange local buckling in wide flange shapes, J. Struct. Div., Proc. ASCE, Vol. 91, No. ST6, pp. 95-116
Livesley, R.K.	1956	The automatic design of structural frames, Quart. J. Mech. Appl. Math., Vol. 9, Part 3
McCart, B.R., Haug, E.J. and Streeter, T.D.	1969	Optimal design of structures with constraints on natural frequency, paper presented at the AIAA Structural Dynamics and Aeroelasticity Space Conference, New Orleans
Marshall, J.	1971	Torsional behaviour of structural rectangular hollow sections, Struct. Eng., Vol. 49, No. 8, pp. 375-79
Martin, H.C.	1966	Introduction to matrix methods of structural analysis, McGraw-Hill, New York

Masur, E.F. 1974 Optimal structural design for a discrete set of available structural members, Comp. Meth. Appl. Mech. Eng., Vol. 3, No. 2, pp. 195-207

Mei, C. 1970 Coupled vibrations of thin walled beams of open section using the finite element method, Int. J. Mech. Sci., Vol. 12, No. 10, pp. 883-91

Moses, F. and Onada, S. 1969 Minimum weight design of structures with application to elastic grillages, Int. J. Num. Meth. Eng., Vol. 1, No. 4, pp. 311-31

Nelder, J.A. and Mead, R. 1965 A simplex method for function minimization, Comp. J., Vol. 7, No. 4, pp. 308-13

Nelson, R.B. and Felton, L.P. 1972 Thin walled beams in frame synthesis, AIAA J., Vol. 10, No. 12, pp. 1565-69

Nethercot, D.A. and Rockey, K.C. 1971 Finite element solutions for the buckling of columns and beams, Int. J. of Mech. Sci., Vol. 13, No. 11, pp. 945-49

Nethercot, D.A. 1974 Torsional rigidity of rolled steel sections, Int. J. Mech. Sci., Vol. 16, No. 8, pp. 565-72

Nitssche, R.N. and Miller, R.E. 1976 (i) Torsion and flexure of closed thin walled beams,
(ii) Stiffness matrices for beam members including warping torsion effects, J. Eng. Mech. Div., Proc. ASCE, Vol. 102, No. EM1, pp. 189-97

Nylander, H. 1956 Torsion, bending and lateral buckling of I-beams, Bull. No. 12, Div. Bldg. Stat. Struct. Eng., Royal Institute of Technology, Stockholm, Sweden

Patterson, O. 1952 Combined bending and torsion of I-beams of mono-symmetrical cross-section, Bull. No. 10, Div. Bldg. Stat. Struct. Eng., Royal Institute of Technology, Stockholm, Sweden

Paz, M. Strehl, P. and Schrader, P.	1976	Computer determination of the shear centre of open and closed sections, Comp. Struct., Vol. 6, pp. 117-25
Pearson, C.	1958	Structural design by high speed computing machines, Proc. ASCE Conf. on Electronic Computation, Kansas City, Missouri
Popelar, C.H.	1969	Dynamic stability of the flexural vibrations of a thin walled beam, Int. J. of Solids Struct., Vol. 5, No. 5, pp. 549-57
Powell, M.J.D.	1964	An efficient method for finding the minimum of a function of several variables without calculating derivatives, Comp. J., Vol. 7, No. 4, pp. 155-62
Przemieniecki, J.S.	1968	Theory of matrix structural analysis, McGraw-Hill, New York
Rajasekharan, S.	1971	Structural Engineering Report No. 34, Department of Civil Engineering, University of Alberta, Edmonton, Canada
Rajasekharan, S. and Murray, D.W.	1973	Coupled local buckling in wide flange beam columns, J. Struct. Div., Proc. ASCE, Vol. 99, No. ST6, pp. 1003-23
Ralston, A.	1965	A first course in numerical analysis, McGraw-Hill, New York
Ralston, A. and Wilf, H.S., Ed. by	1967	Mathematical methods for digital computers, Vol. II, John Wiley, New York
Rao, S.S.	1978	Optimization theory and applications, Wiley Eastern, New Delhi
Rejman, W.	1955	Stability of reinforced concrete retaining walls and abutments, J. Am. Concrete Inst., Vol. 26, No. 10, pp. 1013-23
Renton, J.D.	1967	Buckling of frames composed of thin walled members, 'Thin walled structures', Ed. by Chilver, A.H., John Wiley, New York

Rockey, K.C. and Hill, H., Ed. by — 1969 — Thin walled steel structures: their design and use in building, Symposium at University College of Swansea School of Engineering, 11-14 Sept. 1967, Gordon and Breach, Science Publishers, New York

Rosenbrock, H.H. — 1960 — An automatic method for finding the greatest or least value of a function, Comp. J., Vol. 3, No. 3, pp. 175-84

Schilling, C.G. — 1974 — Optimum properties for I-shaped beams, J. Struct. Div., Proc. ASCE, Vol. 100, No. ST12, pp. 2385-2401

Schmit, L. — 1960 — Structural design by systematic synthesis, Proc. ASCE 2nd Conf. on Electronic Computation, Pittsburgh, Pa.

Shanley, F.R. — 1960 — Weight/strength analysis of aircraft structures, Dover Publications, New York

Sheu, C.Y. — 1968 — Elastic minimum weight design for specified fundamental frequency, Int. J. Solids Struct., Vol. 4, No. 10, pp. 953-58

SP : 6(1) — 1964 — ISI handbook for structural engineers

Spunt, L. — 1971 — Optimum structural design, Prentice-Hall, Englewood Cliffs, New Jersey

Spendley, W., Hert, G.R. and Himsworth, F.R. — 1962 — Sequential application of simplex designs in optimization and evolutionary operation, Technometrics, Vol. 4, No. 4, pp. 441-61

Stiles, J.W. — 1970 — Practical aspects of cantilever retaining wall design, Highway Res. Board Note No. 302, pp. 87-96

Taylor, J.E. — 1968 — Optimum design of a vibrating bar with a specified minimum cross-section, AIAA J., Vol. 6, No. 7, Tech. Notes, pp. 1379-81

Taylor, J.E. and Liu, C.Y. — 1968 — Optimal design of columns, AIAA J., Vol. 6, No. 8, pp. 1497-1502

Terzaghi, K.	1960	Soil mechanics in engineering practice, Asia Publishing House, Bombay
Tezcan, S.S.	1965	Cantilever retaining walls, Proc. Inst. Civil Eng., Part 1, Vol. 68, No. 708, July 1965 and Part 2, Vol. 68, No. 703, Aug. 1965
Thomson, W.T.	1972	Theory of vibration with applications, Prentice-Hall, Englewood Cliffs, New Jersey
Timoshenko, S. and Gere, J.M.	1961	Theory of elastic stability, McGraw-Hill, New York
Tschebotarioff, G.	1951	Soil mechanics, foundations and earth structures, McGraw-Hill, New York
Turner, M.J.	1967	Design of minimum mass structures with specified natural frequencies, AIAA J., Vol. 5, No. 3, pp. 406-12
Vlasov, V.Z.	1959	Thin walled elastic beams, English translation published for U.S. Science Foundation by Israel Program for Scientific Translations, 1961
Williams, F.W. and Wittrick, W.H.	1969	Computational procedures for a matrix analysis of the stability and vibration of the thin flat walled structures in compression, Int. J. Mech. Sci., Vol. 11, No. 12, pp. 979-98
Wittrick, W.H.	1968	General sinusoidal stiffness matrices for buckling and vibration analysis of thin flat walled structures, Int. J. Mech. Sci., Vol. 10, No. 12, pp. 949-66
Wittrick, W.H. and Williams, F.W.	1973	An algorithm for calculating critical buckling loads of elastic structures, J. Struct. Mech., Vol. 1, pp. 497-518
Yura, J.A. and Oral, A.	1972	Web behaviour at points of concentrated loads in steel beams, AISI Proj. No. 157, Report No. 4, Dept. of Civil Eng., Univ. of Texas, Austin, Texas

Zangwill, W.I. 1967 Non-linear programming via penalty functions, Man. Sci., Vol. 13, No. 5, pp. 344-58

Zarghamee, M.S. 1970 Minimum weight design with stability constraints, J. Struct. Div., Proc. ASCE, Vol. 96, No. ST8, pp. 1697-1710

Zbirshowski-Koscia,K. 1967 Thin walled beams: from theory to practice, Crossby Lockwood, London

Zienkiewicz, O.C. 1971 The finite element method in engineering science, McGraw-Hill, New York

Zoutendijk, G. 1960 Methods of feasible directions, Elsevier, Amsterdam.

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Table 5.1

Representative Values of Soil Properties

No.	Type of soil	w_s (kg/m^3)	ϕ (degrees)	p_{per} (kg/m^2)	c (kg/m^2)
1.	Gravel	2080	38	55000	0
2.	Dense sand	1920	33	45000	0
3.	Loose sand	1880	28	30000	0
4.	Silt	1720	22	20000	2000
5.	Clay	1720	18	16000	5000

6.2.1 Optimization of Thin-Walled Sections under Static Constraints

For four different loading conditions considered, the results are presented for different spans and boundary conditions in Tables 6.2 through 6.37.

6.2.1.1 Uniformly distributed load

The results are presented in Tables 6.2 through 6.7.

For fixed support conditions, the optimum weight changes from 9.05 to 25.01 kg/m as the span changes from 3.0 to 6.0 m, with the loading intensity remaining at 1000 kg/m. As the loading intensity is increased from 2000 to 3000 kg/m, the corresponding range of the optimum weights is 13.45-32.32 kg/m and 16.87-38.77 kg/m respectively. In each case the constraint on the normal stress in the beam is seen to be active.

However, in the case of simple support conditions, for a loading intensity of 1000 kg/m, a change in the span from 3.0 to 6.0 m enhances the optimum weight from 12.84 to 29.70 kg/m. In each case, the constraint on the deflection in the beam is found to be active, except for a span of 5.0 m wherein constraints on the normal stress and the deflection are both found to be active, as indicated in Table 6.2. For the loading intensity of 2000 kg/m, the optimum weight ranges from 16.87 to 40.01 kg/m and the

failure of the wall by sliding on the foundation. When the height is 6.0 m, the active constraint is on the minimum stem root thickness to resist bending moment.

When the top width has values of 0.25 and 0.30 m, the respective range of the optimum cost is Rs. 577.87-2171.79 and Rs. 630.70-2197.80, as the wall height varies from 3.0 to 6.0 m. As before, the active constraint is the lower bound on the stem root thickness for smaller wall heights and that on the safety against sliding, for larger heights.

6.2.4.4 Silt

For a top width of 0.20 m and the range of heights from 3.0 to 6.0 m, the optimum cost varies from Rs. 496.81 to Rs. 2157.44. For heights upto 4.0 m, the active constraint is the lower bound on the stem root thickness; between 4.0 to 5.0 m, it is on the safety against sliding while for larger heights, it is the minimum stem root thickness to resist bending moment.

For top widths of 0.25 and 0.30 m, the optimum cost variation is from Rs. 523.92-2185.26 and Rs. 584.50-2193.39 respectively, as the wall height changes from 3.0 to 6.0 m. The active constraint is the lower bound on the stem root thickness for smaller heights and that on the safety against sliding, for larger heights of the wall.

Table 6.7

Optimal Sectional Dimensions

Loading: uniformly distributed

Intensity: 3000 kg/m ; Lateral Eccentricity: nil

Span in m	Cantilever							
	Optimal dimensions in mm						Optimum weight in kg/m	Active constra-int/s
	bf_1	tf_1	bf_2	tf_2	hw	tw		
1.0	78.4	6.7	78.4	6.7	142.0	4.7	13.45	g_{13}^{*}
1.5	100.0	7.8	100.0	7.8	209.0	5.5	21.29	g_{13}
2.0	130.4	8.7	130.4	8.7	259.0	6.2	30.52	g_{13}, g_{15}^{**}

*Maximum normal stress in the beam.

**Maximum deflection in the beam.

Table 6.17

Optimal Sectional Dimensions

Loading: linearly varying

Intensity: 3000-1000 kg/m ; Lateral Eccentricity: nil

Span in m	Cantilever							
	Optimal dimensions in mm						Optimum weight in kg/m	Active constra-int/s
	bf_1	tf_1	bf_2	tf_2	hw	tw		
1.0	63.0	6.5	63.0	6.5	113.0	4.2	10.11	none
1.5	86.8	6.9	86.8	6.9	167.0	5.0	15.92	g_{15}^{**}
2.0	100.0	8.2	100.0	8.2	218.0	5.7	22.66	g_{15}

**Maximum deflection in the beam.

Table 6.27

Optimal Sectional Dimensions

Loading: uniformly distributed

Intensity: 1000 kg/m ; Lateral Eccentricity: 0.2 bf_1

(from Table 6.3)

Span in m	Cantilever							
	Optimal dimensions in mm						Optimum weight in kg/m	Active constra-int/s
	bf_1	tf_1	bf_2	tf_2	hw	tw		
1.0	58.0	6.4	58.0	6.4	108.0	4.1	9.36	g_{13}^*
1.5	86.4	6.9	86.4	6.9	166.0	5.0	15.82	g_{13}
2.0	100.0	8.1	100.0	8.1	216.0	5.6	22.34	g_{13}

*Maximum normal stress in the beam.

Dorn, W.S. and McCracken, D.D.	1972	Numerical methods with Fortran IV case studies, John Wiley, New York
Dupuis, G.	1971	Optimal design of statically determinate beams subjects to displacement and stress constraints, AIAA J., Vol. 9, No. 5, Tech. Notes, pp. 981-84
El Darwish, I. and Johston, B.	1965	Torsion of structural shapes, J. Struct. Div., Proc. ASCE, Vol. 91, No. ST1, pp. 203-27
Felton, L.P. and Nelson, R.B.	1971	Optimized components in frame synthesis, AIAA J., Vol. 9, pp. 1027-31
Fergusson, P.	1965	Reinforced concrete fundamentals, Wiley Eastern Pvt. Ltd., New Delhi
Fiacco, A.V. and McCormick, G.P.	1963	Programming under non-linear constraints by unconstrained minimization: A primal-dual method, Research Analysis Corporation, Bethesda, Md., Tech. Paper No. RAC-TP-96
Fiacco, A.V. and McCormick, G.P.	1964	Computational algorithm for the sequential unconstrained minimization technique for non-linear programming, Man. Sci., Vol. 10, No. 4, pp. 601-17
Fiacco, A.V. and McCormick, G.P.	1968	Non-linear programming: sequential unconstrained minimization techniques John Wiley, New York
Fletcher, R. and Powell, M.J.D.	1963	A rapidly convergent descent method for minimization, Comp. J., Vol. 6, No. 2, pp. 163-68
Fletcher, R. and Reeves, C.M.	1964	Function minimization by conjugate gradients, Comp. J., Vol. 7, No. 2, pp. 149-54
Fox, R.L.	1971	Optimization methods for engineering design, Addison-Wesley, Reading, Mass.
Fried, I.	1969	Gradient methods for finite element eigen problems, AIAA J., Vol. 7, No. 4, Tech. Notes, pp. 739-41